Quantifying the performance of approximate teleportation and quantum error correction via symmetric two-PPT-extendibility

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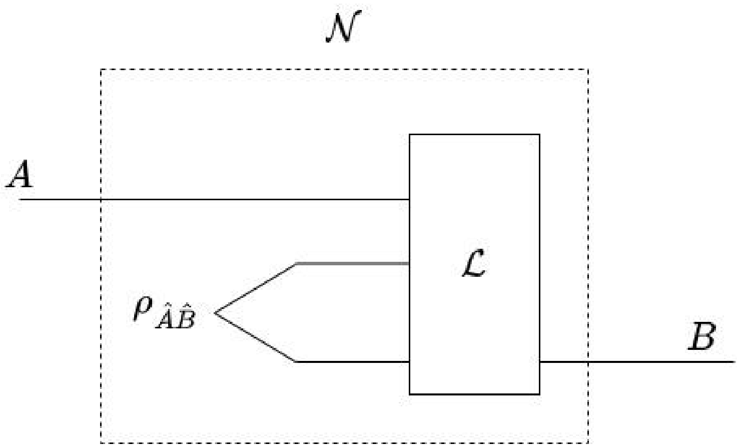
Outline

- Approximate Teleportation
- Semi-Definite Program (SDP) with two-PPT-extendibility constraints
- Symmetries to simplify the SDP
- Connection to Approximate Quantum Error Correction

Approximate teleportation protocol



- Alice and Bob share a known resource state $\rho_{\hat{A}\hat{B}}$
- Compare the channel $\mathcal{N}_{A\to B}$ to an identity channel
- Demand $\mathcal{L}_{A\hat{A}\hat{B}\to B}$ to be a 1-way LOCC channel
- Quantify error in simulation by diamond norm

$$\left\| \mathcal{L}_{A\hat{A}\hat{B}\to B} \circ \mathcal{A}^{\rho}_{\hat{A}\hat{B}} - \mathrm{id}_{A\to B} \right\|_{\diamond}$$

Semi-Definite Program with Two-PPT-extendibility

Two-PPT-extendibility

Problem

- It is difficult to optimize over the set of 1-Way LOCC channels.

Solution

- Find a set of channels that can be characterized by semi-definite constraints
  - Demand the channel to be two-extendible
  - Demand the extension to be PPT-preserving channel
- Optimize over the larger set to obtain lower bounds on the exact simulation error

SDP for Diamond norm

- Diamond distance : $$\|\mathcal{N}_{A\to B} - \mathcal{M}_{A\to B}\|_\diamond = \sup_{\psi_{RA}} \|\mathcal{N}(\psi_{RA}) - \mathcal{M}(\psi_{RA})\|_1$$

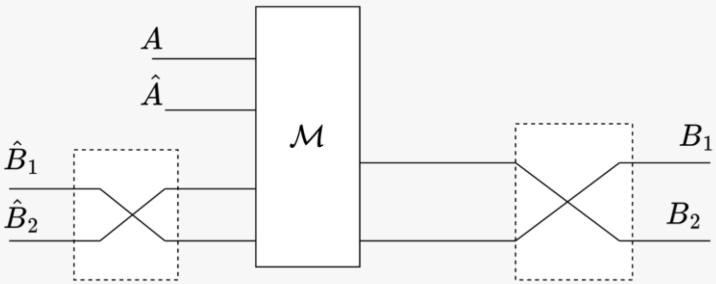
- Diamond norm can be calculated using a semi-definite program

$$\frac{1}{2}\|\mathcal{N}_{A\to B} - \mathcal{M}_{A\to B}\|_\diamond = \inf_{\mu\geq 0, Z_{AB}\geq 0} \mu$$
$$\mu I_A \geq Z_A$$
$$Z_{AB} \geq \Gamma^{\mathcal{N}}_{AB} - \Gamma^{\mathcal{M}}_{AB}$$

- Need to know the Choi operators of the two quantum channels

Two-extendibility conditions

Permutation Covariance

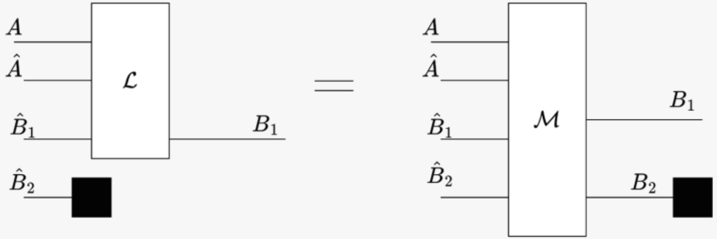


Channel is unchanged under this action

Semi-definite constraint :

$$(\mathcal{F}_{\hat{B}_1\hat{B}_2} \otimes \mathcal{F}_{B_1B_2}) M_{A\hat{A}\hat{B}_1B_1\hat{B}_2B_2} = M_{A\hat{A}\hat{B}_1B_1\hat{B}_2B_2}$$

Non-signaling

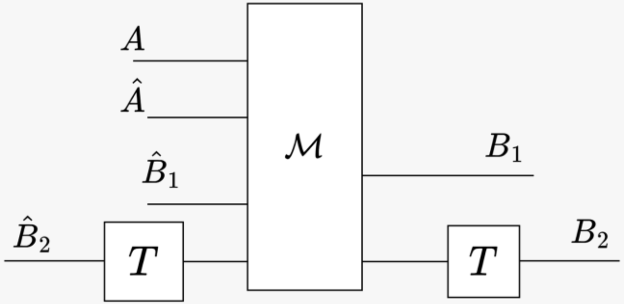


B_1 or B_2 cannot signal back to other parties

Semi-definite constraint :

$$\mathrm{Tr}_{B_2}[M_{A\hat{A}\hat{B}_1B_1\hat{B}_2B_2}] = \frac{M_{A\hat{A}\hat{B}_1B_1}}{d_{\hat{B}}} \otimes I_{\hat{B}_2}$$

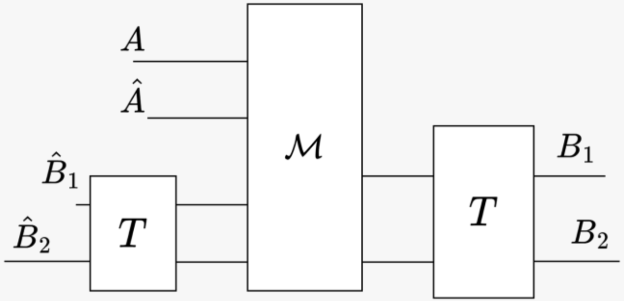
PPT conditions with the extension



This is also a CPTP map

Semi-definite constraint :

$$T_{\hat{B}_2 B_2}(M_{A\hat{A}\hat{B}_1 B_1 \hat{B}_2 B_2}) \geq 0$$



This is also a CPTP map

Semi-definite constraint :

$$T_{A\hat{A}}(M_{A\hat{A}\hat{B}_1 B_1 \hat{B}_2 B_2}) \geq 0$$

Diamond norm SDP with 2-PPT-extendibility

$$e_{2\mathrm{PE}}(\rho_{\hat{A}\hat{B}}) = \inf_{\substack{\mu\geq 0, Z_{AB}\geq 0,\\ M_{A\hat{A}\hat{B}_1B_1\hat{B}_2B_2}\geq 0}} \mu,$$

$$\mu I_A \geq Z_A,$$

$$Z_{AB} \geq \Gamma^{\mathrm{id}}_{AB} - \mathrm{Tr}_{\hat{A}\hat{B}_1}\left[T_{\hat{A}\hat{B}_1}(\rho_{\hat{A}\hat{B}_1})\frac{M_{A\hat{A}\hat{B}_1B_1}}{d_{\hat{B}}}\right]$$

Diamond norm SDP

$$\mathrm{Tr}_{B_1B_2}[M_{A\hat{A}\hat{B}_1B_1\hat{B}_2B_2}] = I_{A\hat{A}\hat{B}_1\hat{B}_2},$$

→ Trace preserving

$$(\mathcal{F}_{\hat{B}_1\hat{B}_2}\otimes\mathcal{F}_{B_1B_2})(M_{A\hat{A}\hat{B}_1B_1\hat{B}_2B_2}) = M_{A\hat{A}\hat{B}_1B_1\hat{B}_2B_2},$$

→ Permutation covariance

$$\mathrm{Tr}_{B_2}[M_{A\hat{A}\hat{B}_1B_1\hat{B}_2B_2}] = \frac{M_{A\hat{A}\hat{B}_1B_1}}{d_{\hat{B}}}\otimes I_{\hat{B}_2},$$

→ Non-signaling

$$T_{A\hat{A}}(M_{A\hat{A}\hat{B}_1B_1\hat{B}_2B_2}) \geq 0,$$

$$T_{\hat{B}_2B_2}(M_{A\hat{A}\hat{B}_1B_1\hat{B}_2B_2}) \geq 0.$$

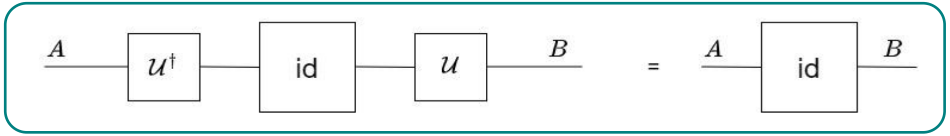
PPT-preserving

Symmetries to simplify the SDP

Symmetry in the problem

Unitary invariance of identity channel

$$\mathcal{U}_B \circ \mathrm{id}_{A\to B} \circ \mathcal{U}_A^\dagger = \mathrm{id}_{A\to B}$$



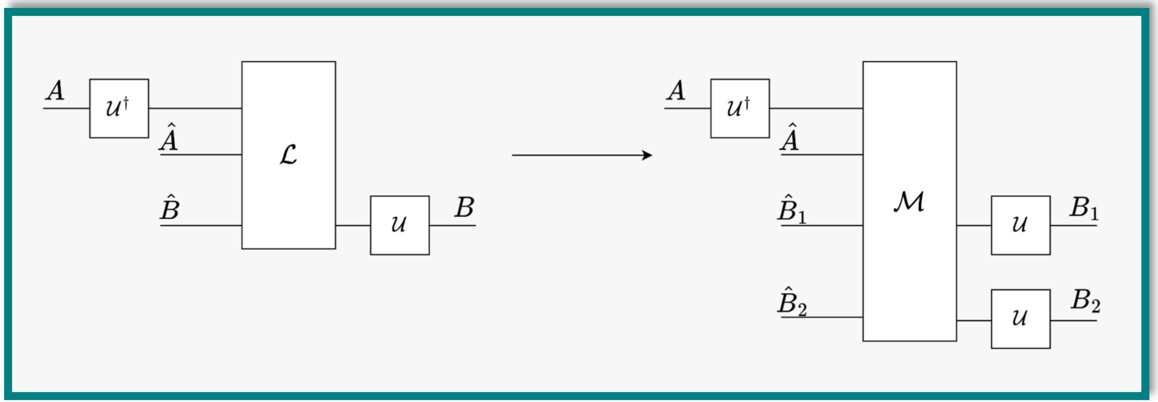
Unitary invariance of diamond norm

$$\begin{aligned}\|\mathcal{L}_{A\hat{A}\hat{B}\to B} \circ \mathcal{A}^\rho_{\hat{A}\hat{B}} - \mathrm{id}_{A\to B}\|_\diamond &= \| \mathcal{U}_B \circ \mathcal{L}_{A\hat{A}\hat{B}\to B} \circ \mathcal{A}^\rho_{\hat{A}\hat{B}} \circ \mathcal{U}_A^\dagger - \mathrm{id}_{A\to B}\|_\diamond \\ &= \int dU \; \| \mathcal{U}_B \circ \mathcal{L}_{A\hat{A}\hat{B}\to B} \circ \mathcal{U}_A^\dagger \circ \mathcal{A}^\rho_{\hat{A}\hat{B}} - \mathrm{id}_{A\to B}\|_\diamond \\ &\geq \| \int dU \; (\mathcal{U}_B \circ \mathcal{L}_{A\hat{A}\hat{B}\to B} \circ \mathcal{U}_A^\dagger) \circ \mathcal{A}^\rho_{\hat{A}\hat{B}} - \mathrm{id}_{A\to B}\|_\diamond\end{aligned}$$

Optimal channel :

$$\mathcal{L}_{A\hat{A}\hat{B}\to B} = \int dU \; \mathcal{U}_B \circ \mathcal{L}_{A\hat{A}\hat{B}\to B} \circ \mathcal{U}_A^\dagger$$

Two-extension of twirled channels



Action on Choi operator :

$$\tilde{L}_{A\hat{A}\hat{B}B} = \widetilde{\mathcal{T}}_{AB}(L_{A\hat{A}\hat{B}B}) = \int dU \; (\bar{\mathcal{U}}_A \otimes \mathcal{U}_B)(L_{A\hat{A}\hat{B}B})$$

↓

$$\widetilde{M}_{A\hat{A}\hat{B}_1\hat{B}_2B_1B_2} = \widetilde{\mathcal{T}}_{AB_1B_2}(M_{A\hat{A}\hat{B}_1\hat{B}_2B_1B_2}) = \int dU \; (\bar{\mathcal{U}}_A \otimes \mathcal{U}_{B_1} \otimes \mathcal{U}_{B_2})(M_{A\hat{A}\hat{B}_1\hat{B}_2B_1B_2})$$

Simplifying the twirled channel

Decomposing the twirled Choi operator

$$\widetilde{M}_{A\hat{A}\hat{B}_1\hat{B}_2B_1B_2} = \sum_{i\in\{+,-,0,1,2,3\}} M^i_{\hat{A}\hat{B}_1\hat{B}_2} \otimes \frac{S^i_{AB_1B_2}}{\mathrm{Tr}[S^i_{AB_1B_2}]}$$

The S matrices :

Orthogonal projectors: $S^+_{AB_1B_2}$, $S^-_{AB_1B_2}$, $S^0_{AB_1B_2}$

$$S^+_{AB_1B_2} \quad S^-_{AB_1B_2} \quad S^0_{AB_1B_2} \quad S^1_{AB_1B_2} \quad S^2_{AB_1B_2} \quad S^3_{AB_1B_2}$$

Pauli like matrices: $S^0_{AB_1B_2}$, $S^1_{AB_1B_2}$, $S^2_{AB_1B_2}$, $S^3_{AB_1B_2}$

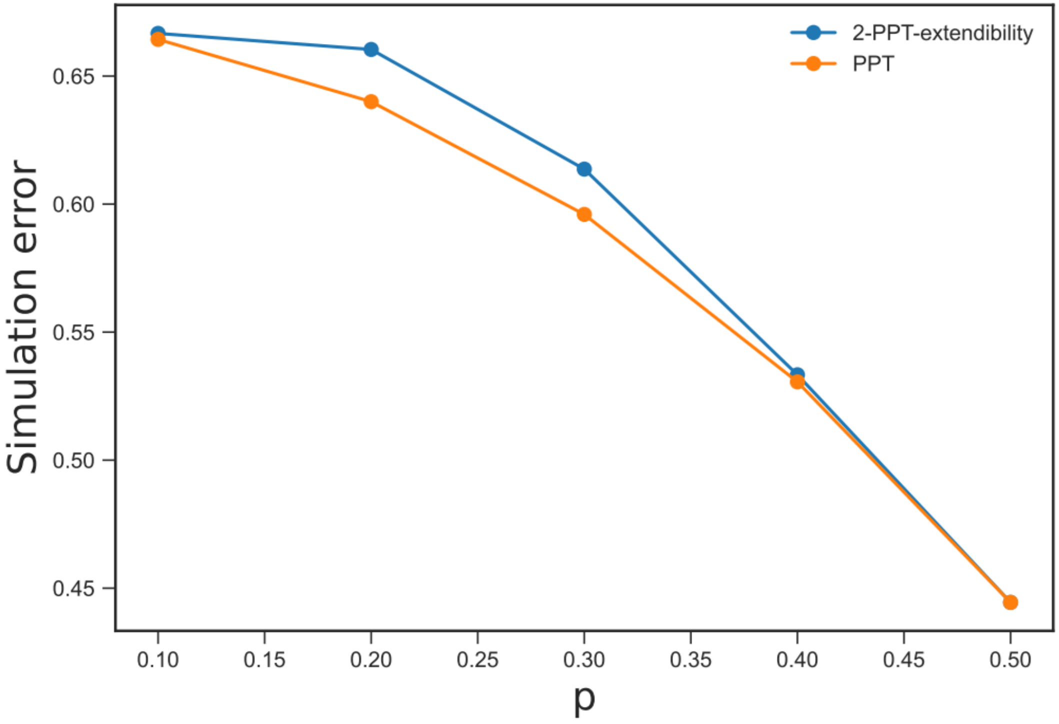
Final SDP depends on 6 variables : $M^+_{\hat{A}\hat{B}_1\hat{B}_2}, \; M^-_{\hat{A}\hat{B}_1\hat{B}_2}, \; M^0_{\hat{A}\hat{B}_1\hat{B}_2}, \; M^1_{\hat{A}\hat{B}_1\hat{B}_2}, \; M^2_{\hat{A}\hat{B}_1\hat{B}_2}, \; M^3_{\hat{A}\hat{B}_1\hat{B}_2}$

Special mixed state as the shared resource state

- Resource state of the form

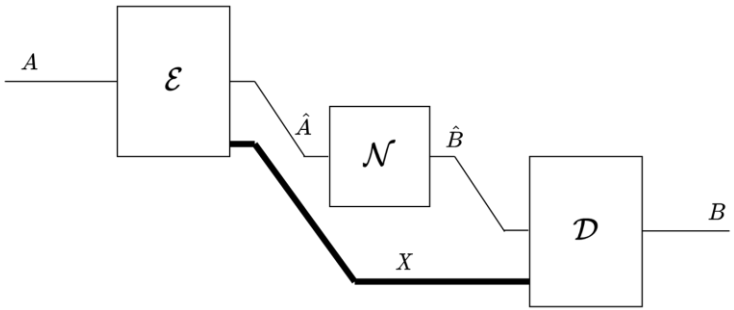
  $p\ \Phi_{\hat{A}\hat{B}} + (1-p)\pi_{\hat{A}} \otimes \sigma_{\hat{B}}$

- Comparison between lower bounds obtained from Two-PPT-extendibility constraints vs bounds from only PPT-preserving constraints



Connection to Approximate Quantum Error Correction

Approximate error correction

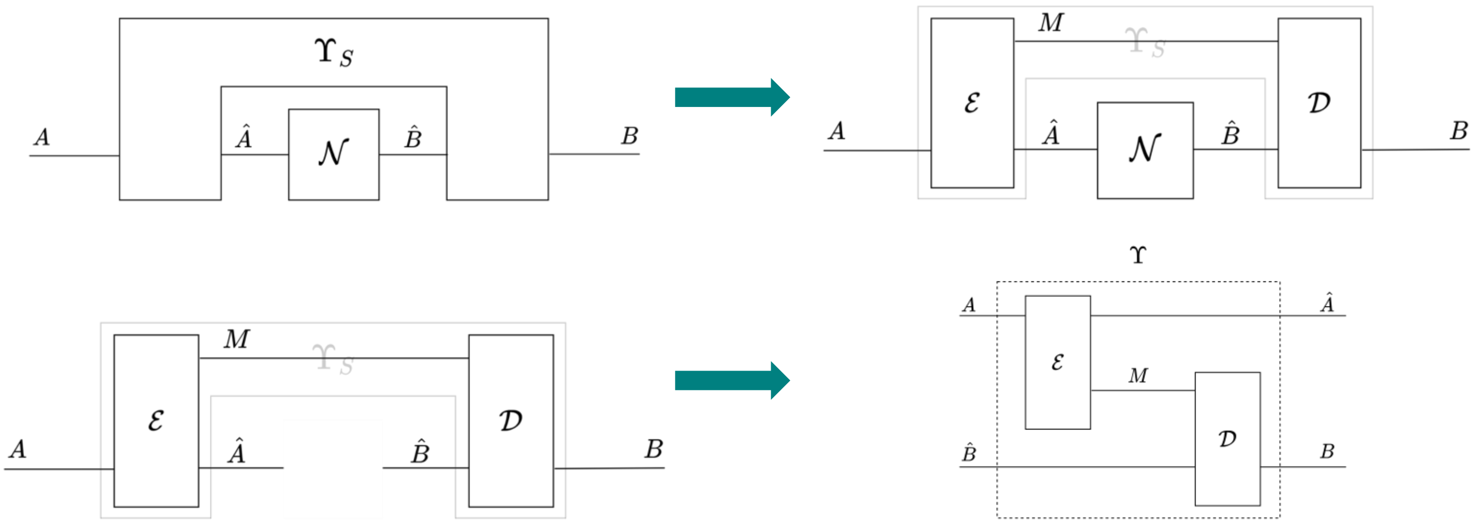


- A known quantum channel $\mathcal{N}_{\hat{A}\to\hat{B}}$ exists between Alice and Bob.
- Simulate and identity channel between Alice and Bob using only the channel assisted with 1-way LOCC.
- Quantify the simulation error using diamond norm

$$\left\|\mathcal{D}_{X\hat{B}\to B} \circ \mathcal{N}_{\hat{A}\to\hat{B}} \circ \mathcal{E}_{A\to\hat{A}X} - \mathrm{id}_{A\to B}\right\|_{\diamond}$$

Using the language of superchannels

- Superchannels are general quantum channel ⟶ quantum channel maps.
- Every superchannel has a unique CPTP map associated with it



Two-PPT-extendible superchannels

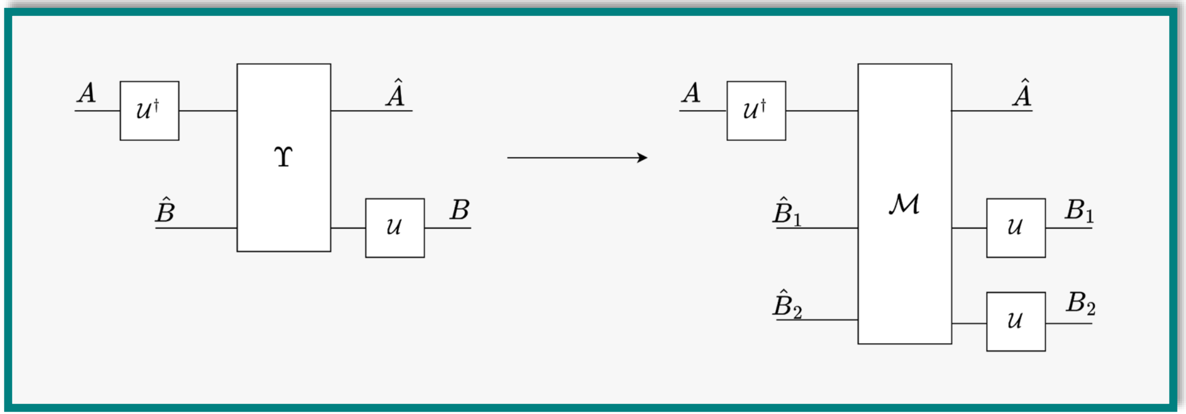
- $\Upsilon^S_{(\hat{A}\to\hat{B})\to(A\to B)}$ is a 1-Way LOCC superchannel
  $\equiv$ $\Upsilon_{A\hat{B}\to\hat{A}B}$ is a 1-Way LOCC channel
  $\equiv$ approx. quantum error correction assisted by 1-Way LOCC

- Optimize over Two-PPT-extendible superchannels to get lower bound on simulation error

Twirled superchannel is optimal

- Unitary invariance of identity channel and diamond norm imply twirled superchannel is optimal.

- Unique CPTP map of twirled and extended superchannel



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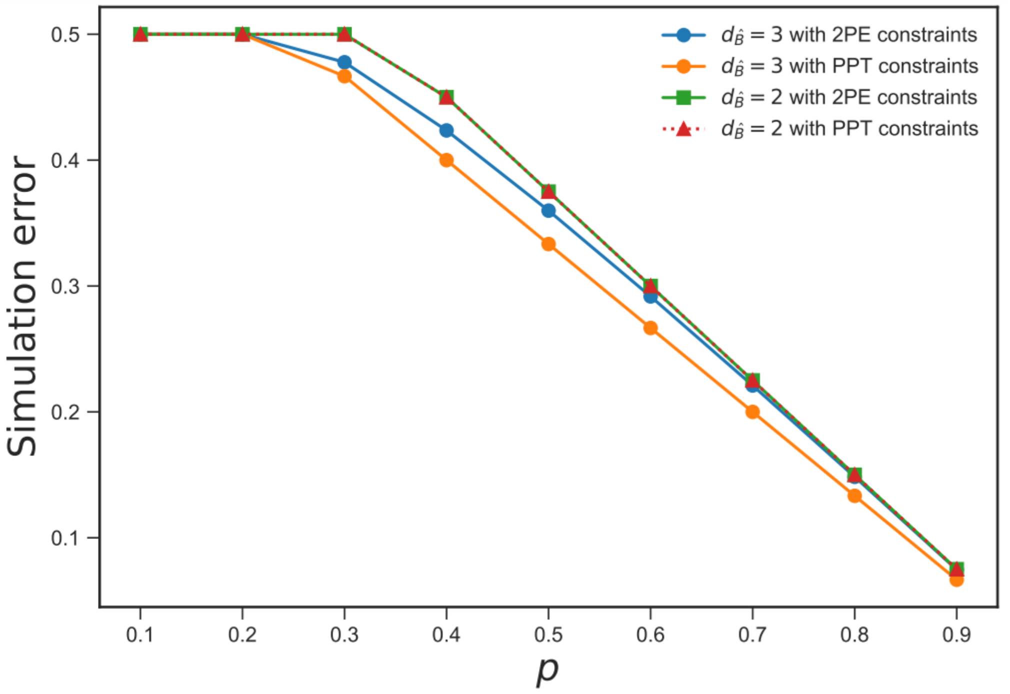
Depolarizing channel as resource channel

- Depolarizing channel used to simulate single qubit identity channel

- Choi state: $p\,\Phi_{\hat{A}\hat{B}} + (1-p)\pi_{\hat{A}\hat{B}}$

- Comparison between qubit and qutrit depolarizing channels, with PPT as well as two-PPT-extendibility constraints



Summary

- We used two-PPT-extendibility conditions to obtain lower bounds on simulation error in approximate teleportation and error correction.

- Symmetry of identity channel significantly reduces computational cost.

- Two-PPT-extendibility conditions give stronger bounds than only PPT conditions when simulating 1-way LOCC channels for certain resources.

Future works

- Systematic analysis of symmetry conditions to generalize for k-PPT-extendibility to obtain stronger bounds.

- Identify class of resource states/channels that saturate the bounds for k-PPT-extendibility conditions.

- Find other semi-definite tightenings of 1-way LOCC channels to get closer to the actual errors

Thank you!