Quantum error correction meets continuous symmetries: fundamental trade-offs and case studies

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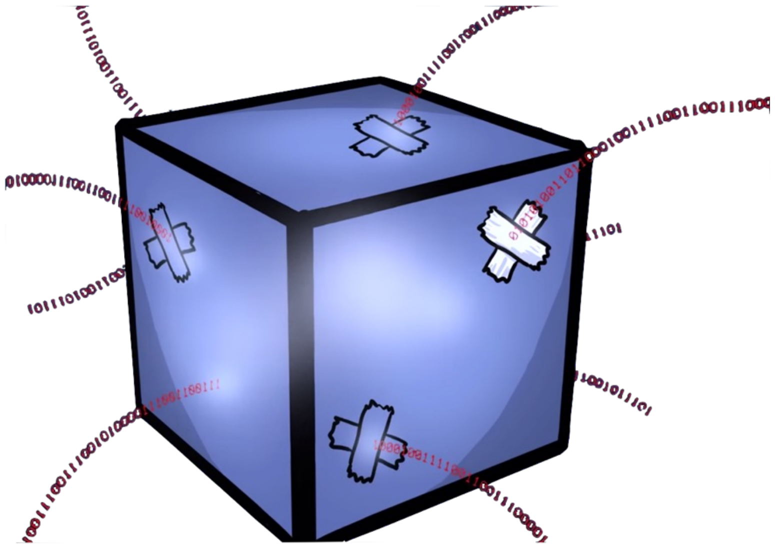
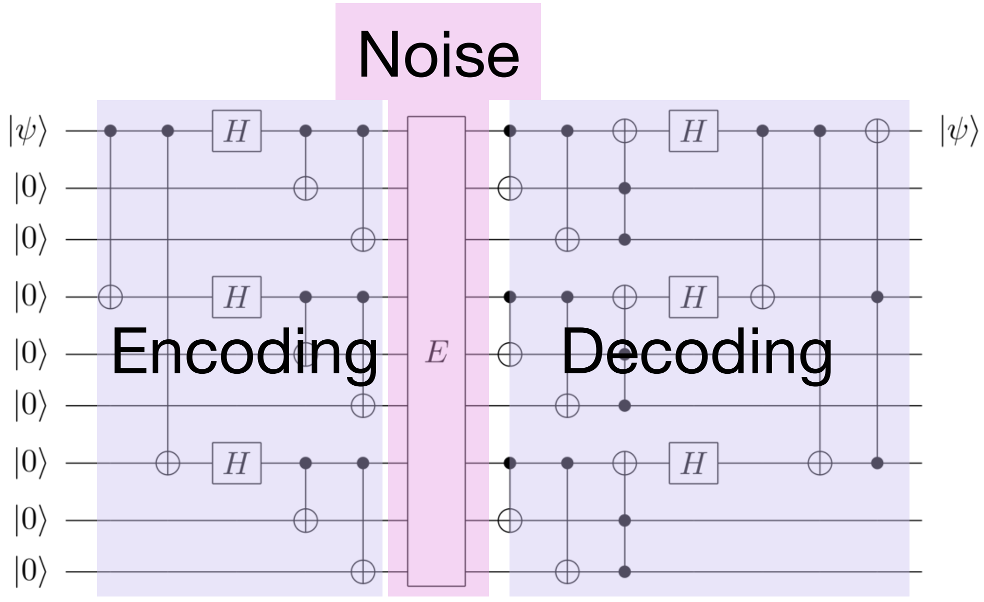
Noise is one of the biggest enemies of quantum computers.

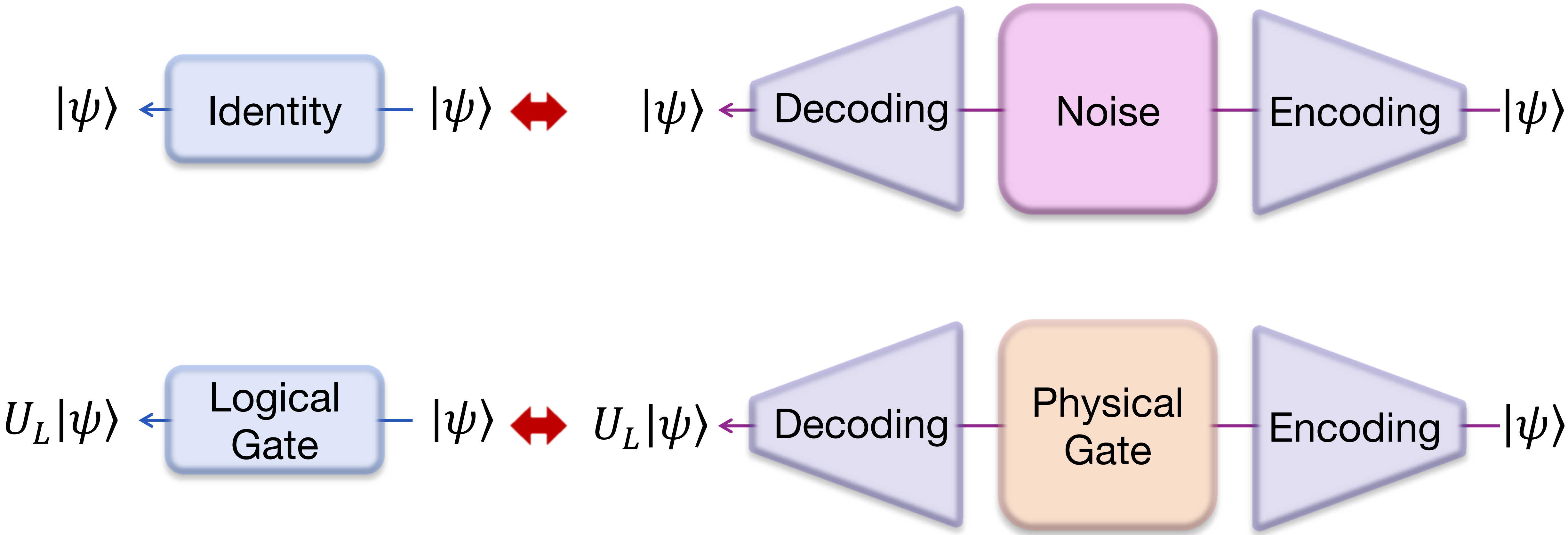




9-qubit Shor code

Quantum error correction protects quantum information from noise, where logical qubits are encoded in a large number of physical qubits.

Quantum error correction (QEC):

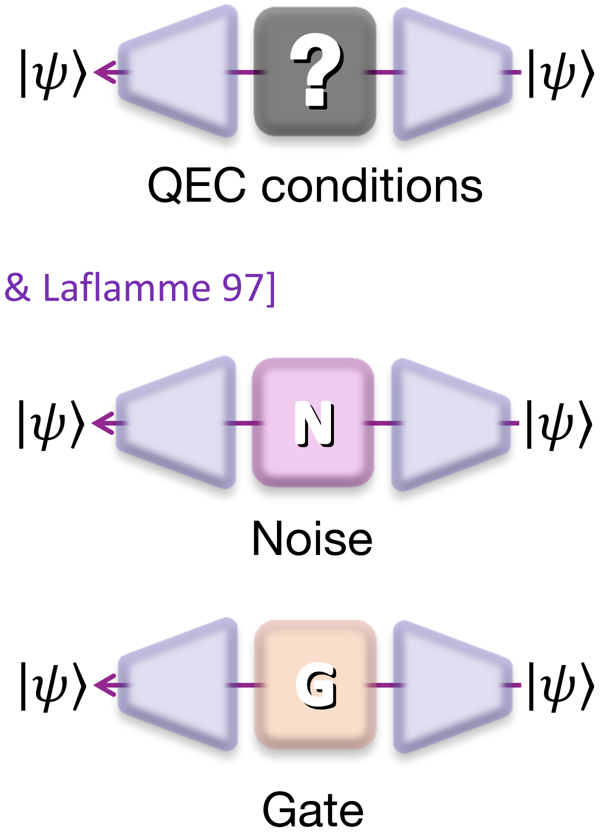


Eastin-Knill Theorem

$$U_{S_1} \otimes U_{S_2} \otimes \cdots \otimes U_{S_n} \Leftrightarrow U_L$$

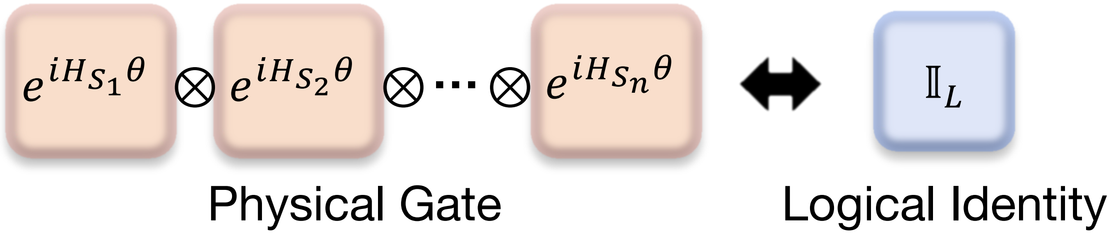
Physical Gate — Logical Gate

For any quantum code that corrects **local errors**, the set of **transversal logical gates** is not universal. [Eastin & Knill 09]



- A noise $\mathcal{N} = \sum_i K_i(\cdot)K_i^\dagger$ is correctable, if and only if $PK_i^\dagger K_j P \propto P$ where P is the code projector. [Bennett *et al.* 96, Knill & Laflamme 97]
- An error-correcting code corrects all local errors, then $PEP \propto P$ for local operators E.
- $\{e^{-iH\theta}\}$, for any local H acts as the identity in the code subspace.
- The connected component of the identity in the group of transversal logical gates acts as the identity.

Eastin-Knill Theorem



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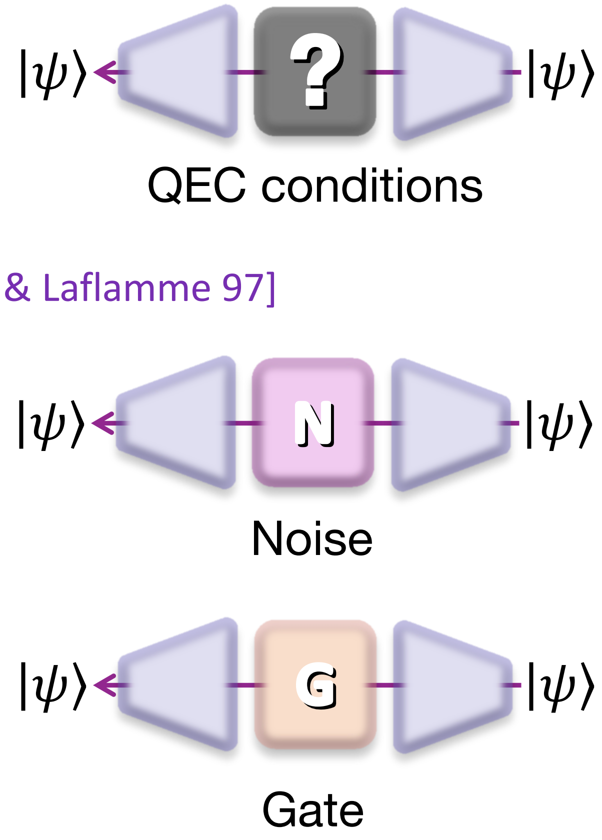
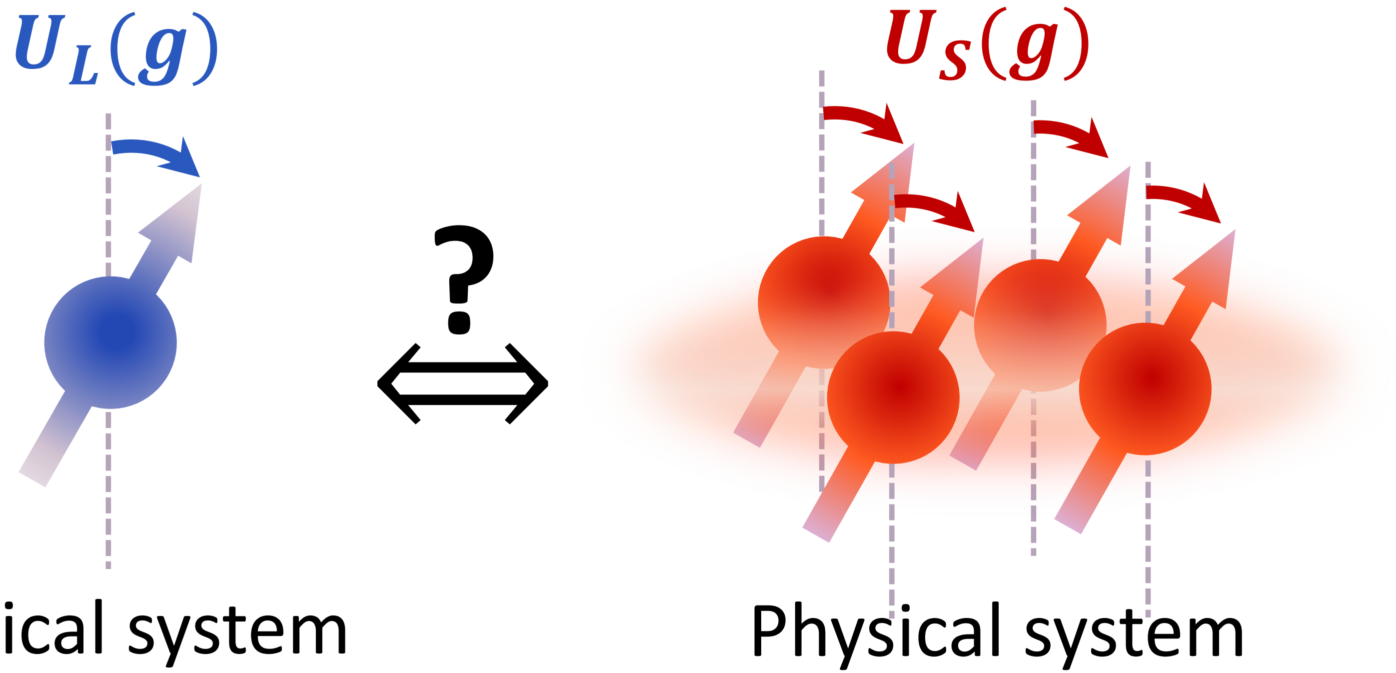
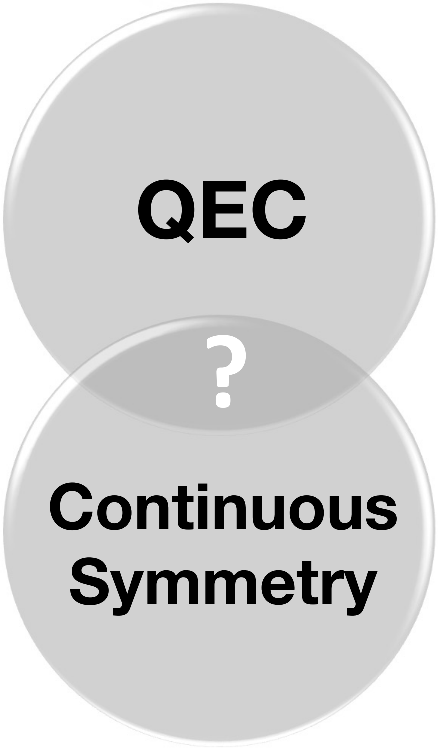
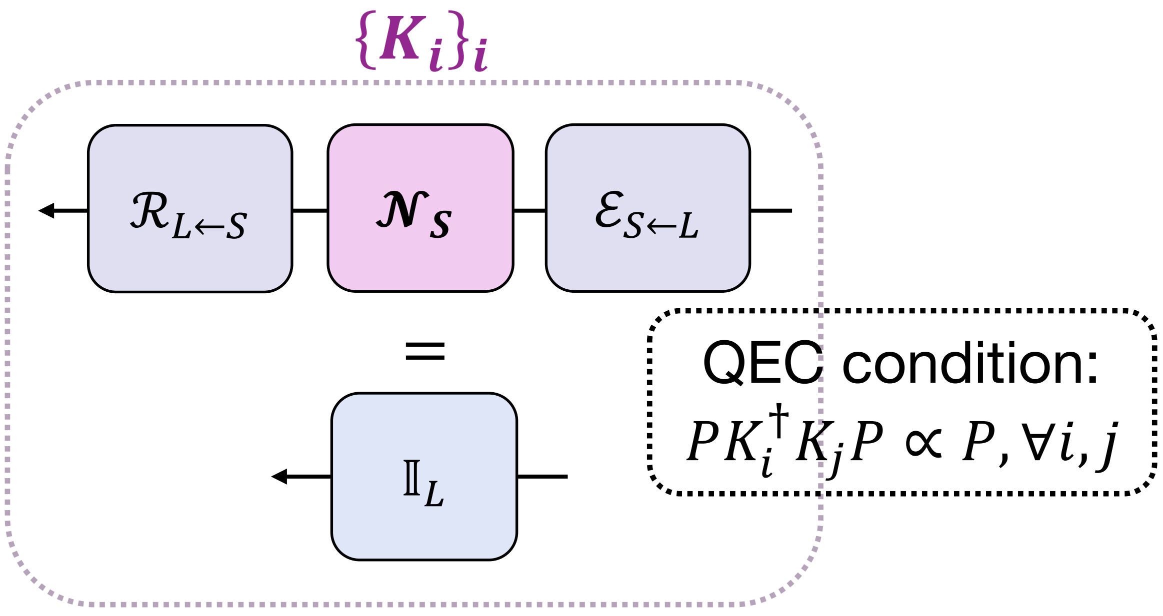


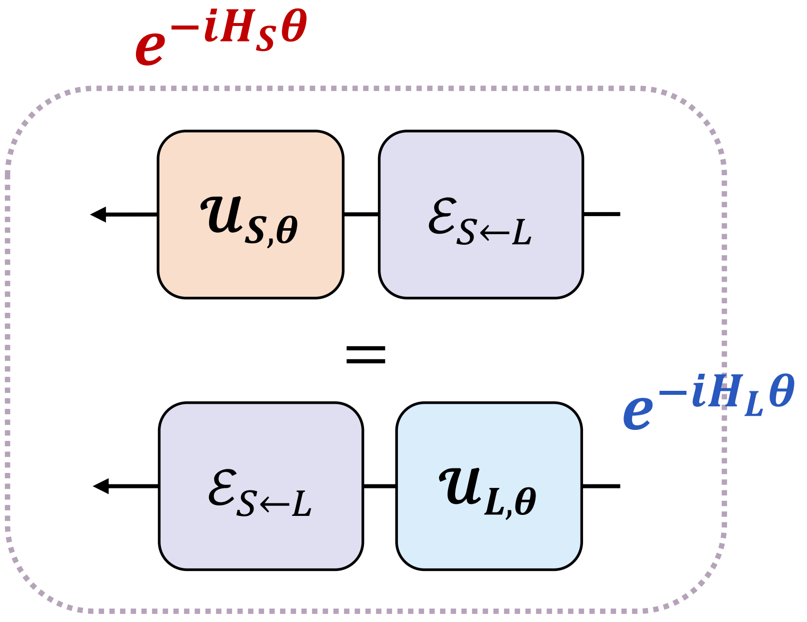
Quantum error correction **with symmetries**





QEC vs. Continuous symmetries

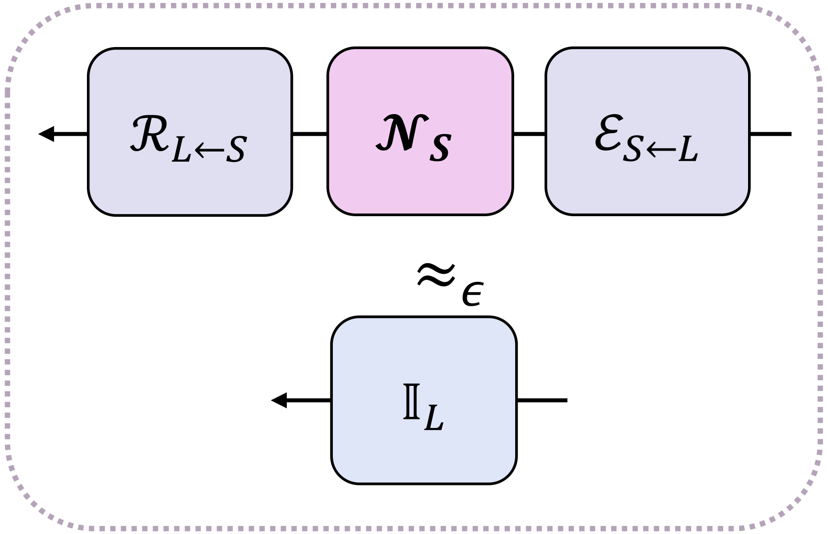




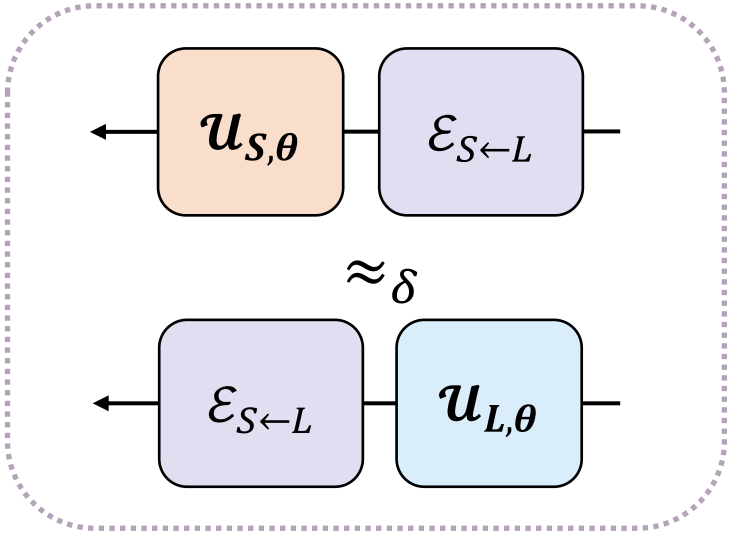
When $H_S \in \mathrm{span}\{K_i^\dagger K_j\}$, H_L acts as the identity.

(the ~~Hamiltonian~~ Charge-in-Kraus-Span (HKS) condition)

Approximate QEC vs. Approximate symmetries



Approximate QEC



Approximate U(1) covariance

ϵ: Purified distance (Minimize over $\mathcal{R}$)

ϵ: Distance measure between $\mathcal{R}_{L\leftarrow S}\circ \mathcal{N}_S \circ \mathcal{E}_{S\leftarrow L}$ and $\mathbb{I}_L$

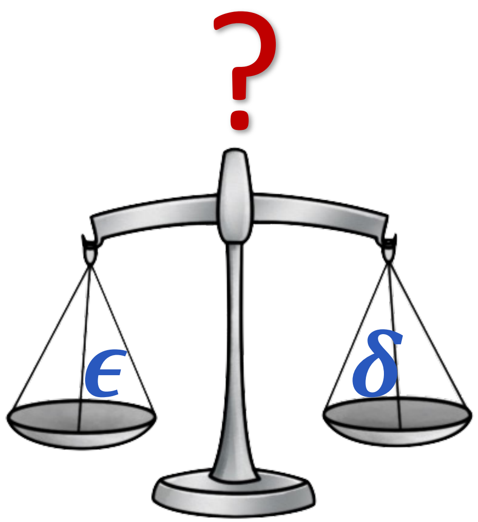
δ: Distance measure between $\mathcal{U}_{S,\theta} \circ \mathcal{E}_{S\leftarrow L}$ and $\mathcal{E}_{S\leftarrow L}\circ \mathcal{U}_{L,\theta}$

3 types of δ: **Global covariance** (Maximize over θ); Local covariance (Derivative w.r.t. θ); Charge conservation

Main results

- Various trade-off relations between QEC and covariance.

$$\delta \gtrsim \sqrt{\frac{\Delta H_L - 2\epsilon\mathfrak{J}(\mathcal{N}_S, H_S)}{\Delta H_S}}, \quad \epsilon + \delta \gtrsim \frac{\Delta H_L}{\sqrt{4\mathfrak{F}(\mathcal{N}_S, H_S)}}, \quad \cdots$$



(blue): Strengths of Hamiltonians. $\lambda_{\max}(H) - \lambda_{\min}(H)$.
(red): Related to the HKS condition.
(yellow): Regularized QFI of $\mathcal{N}_S \circ \mathcal{U}_{S,\theta}$

- Code examples that nearly attain the bounds.

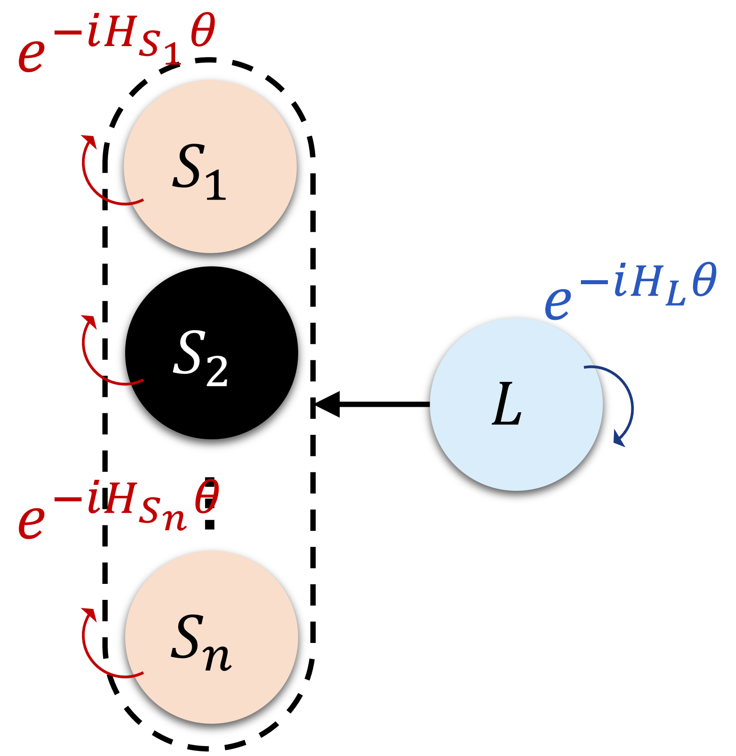
Exactly Covariant Codes:

- Lower bounds on the QEC inaccuracy ϵ when $\delta = 0$.
- Connection to quantum metrology & quantum resource theory.

[Faist *et al.* PRX'20]
[Woods & Alhambra, Quantum'20]
[Kubica & Demkowicz-Dobrzański, PRL'20]
[SZ *et al.* Quantum'21]
[Yang *et al.* PRR'22]

Behavior of the trade-off relations

$$\delta \gtrsim \sqrt{\frac{\Delta H_L - 2\epsilon\, O(n)}{O(n)}} \qquad \epsilon + \delta \gtrsim \frac{\Delta H_L}{\sqrt{4\, O(n^2)}}$$



For sufficiently small δ,

$$\epsilon = \Omega\left(\frac{1}{n}\right)$$

For sufficiently small ϵ,

$$\delta = \Omega\left(\frac{1}{\sqrt{n}}\right) \qquad \delta = \Omega\left(\frac{1}{n}\right)$$

Random local noise:

$$\mathcal{N}_{S,\theta} = \sum_{l=1}^{n} \frac{1}{n} \mathcal{N}_{S_l,\theta}$$

Noise acts on one subsystem chosen uniformly at random.

Behavior of the trade-off relations

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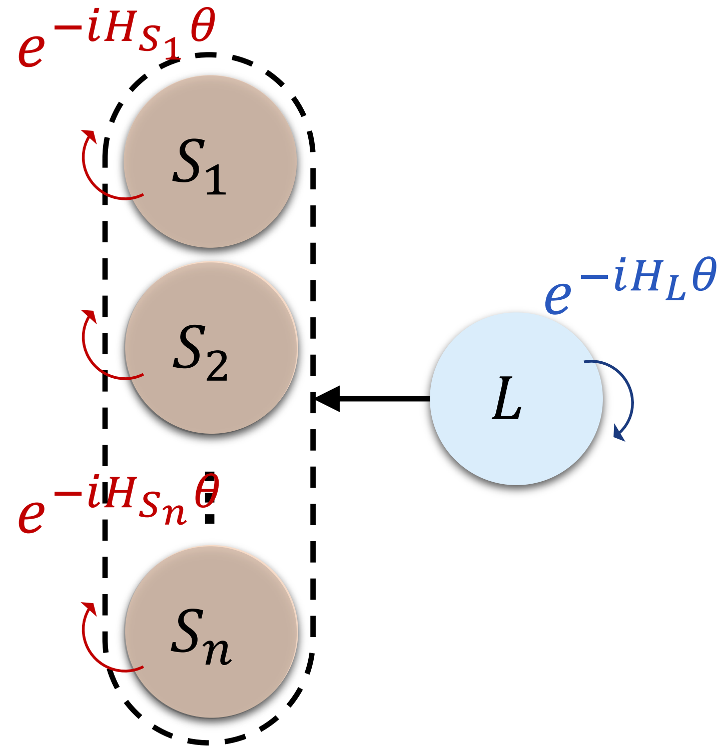

For sufficiently small δ,

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For sufficiently small ϵ,

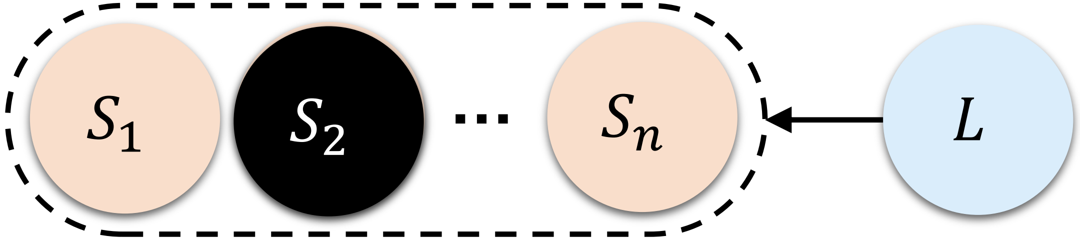
$$\delta = \Omega\left(\frac{1}{\sqrt{n}}\right) \; -$$

I.I.D. noise:

$$\mathcal{N}_{S,\theta} = \otimes_{l=1}^{n} \mathcal{N}_{S_l,\theta}$$

Noise acts on every subsystem with a fixed probability.

Limitation on transversal logical gates



$$\exp\left(\frac{-i2\pi H_{S_1}}{D}\right) \otimes \cdots \exp\left(\frac{-i2\pi H_{S_n}}{D}\right) = \exp\left(\frac{-i2\pi H_L}{D}\right)$$

$$\exp(i\theta H_{S_1}) \otimes \cdots \exp(i\theta H_{S_1}) \approx_\delta \exp(i\theta H_L)$$

$$\Longrightarrow$$

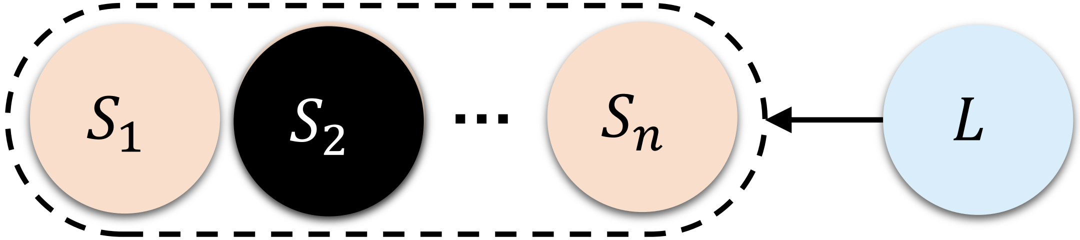
$$\delta \cdot D = O(n),$$

$$D = O(n^{3/2}).$$

$$\delta = \Omega(1/\sqrt{n})$$

Consider a QEC code that corrects local errors and admits a transversal implementation $V_S = \otimes_{l=1}^{n} e^{-i2\pi H_{S_l}/D}$ of the logical gate $V_L = e^{-i2\pi H_L/D}$, where D is an integer, and $H_{L,S}$ have integer eigenvalues and have constant scalings.

Limitation on transversal logical gates



The $O(\log n)$-th level Clifford hierarchy for stabilizer codes

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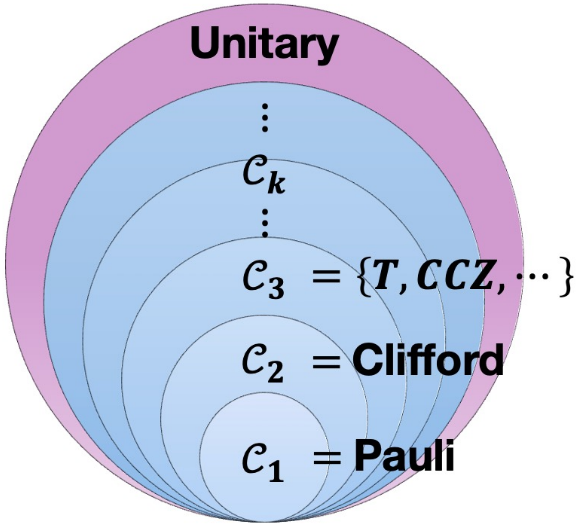

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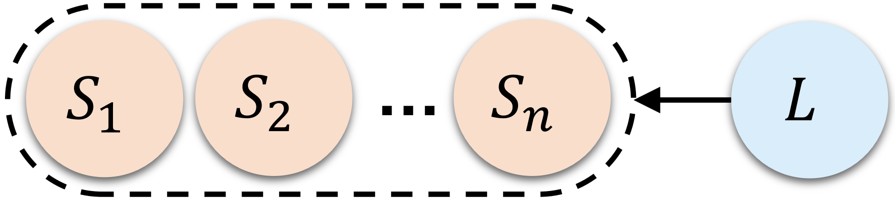
Example: Quantum Reed-Muller code



$$\exp\left(\frac{i\pi Z_{S_1}}{2^{t-1}}\right) \quad \cdots \quad \exp\left(\frac{i\pi Z_{S_n}}{2^{t-1}}\right) = \exp\left(\frac{-i\pi Z_L}{2^{t-1}}\right)$$

Consider the $[[n = 2^t - 1, 1, 3]]$ quantum Reed-Muller code:

$$|\mathfrak{c}_0\rangle = \frac{1}{\sqrt{2^t}}\left(\sum_{\mathbf{x}\in \overline{\mathrm{RM}(1,t)}} |\mathbf{x}\rangle\right), \quad |\mathfrak{c}_1\rangle = \frac{1}{\sqrt{2^t}}\left(\sum_{\mathbf{x}\in \overline{\mathrm{RM}(1,t)}} |\mathbf{1} + \mathbf{x}\rangle\right).$$

The code is exactly error-correcting against single-qubit errors, and is approximately covariant:

$$\epsilon = 0, \qquad \delta \approx \frac{2}{\sqrt{n}} \gtrsim \frac{1}{\sqrt{n}}.$$

$\mathbf{0}$	0	0	0	0	0	0	0	0	$|\mathfrak{c}_0\rangle$
v_3	0	0	0	0	1	1	1	1	
v_2	0	0	1	1	0	0	1	1	
v_1	0	1	0	1	0	1	0	1	
$v_2 + v_3$	0	0	1	1	1	1	0	0	
$v_1 + v_3$	0	1	0	1	1	0	1	0	
$v_1 + v_2$	0	1	1	0	0	1	1	0	
$v_1 + v_2 + v_3$	0	1	1	0	1	0	0	1	
$\mathbf{1}$	1	1	1	1	1	1	1	1	
$\mathbf{1} + v_3$	1	1	1	1	0	0	0	0	
$\mathbf{1} + v_2$	1	1	0	0	1	1	0	0	
$\mathbf{1} + v_1$	1	0	1	0	1	0	1	0	
$\mathbf{1} + v_2 + v_3$	1	1	0	0	0	0	1	1	
$\mathbf{1} + v_1 + v_3$	1	0	1	0	0	1	0	1	
$\mathbf{1} + v_1 + v_2$	1	0	0	1	1	0	0	1	
$\mathbf{1} + v_1 + v_2 + v_3$	1	0	0	1	0	1	1	0	

$|\mathfrak{c}_1\rangle = X^{\otimes n}|\mathfrak{c}_0\rangle$

$t = 3$

Example: Modified thermodynamic code

Exact covariance $\delta = 0$

Exact QEC $\epsilon = 0$

Parameter $q \in [0,1]$ — 0 to 1

Thermodynamic code ($q = 0$),
$|m\rangle_{\text{Dicke}} \propto \sum_{\boldsymbol{j}\in\{0,1\}^n, H_S|\boldsymbol{j}\rangle = m|\boldsymbol{j}\rangle} |j\rangle$.

[Brandao et al. PRL'19, Faist *et al.* PRX'21]

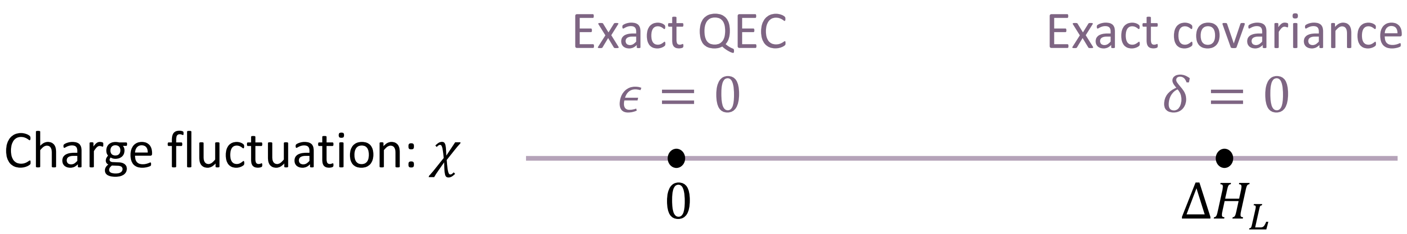
Consider a spin chain where the total charge $H_S = -\sum_{l=1}^{n} Z_{S_l}$.

$$\left|c_0^q\right\rangle \propto \sqrt{n}|m\rangle_{\text{Dicke}} + \sqrt{qm}|-n\rangle_{\text{Dicke}}, \qquad \left|c_1^q\right\rangle \propto \sqrt{n}|-m\rangle_{\text{Dicke}} + \sqrt{qm}|n\rangle_{\text{Dicke}}.$$

The code transits smoothly from an exactly covariant code to an exactly error-correcting code when q increases from 0 to 1:

$$\epsilon \approx \frac{(1-q)m}{2n} \gtrsim \frac{(1-4q)m}{2n}, \qquad \delta \approx \frac{\sqrt{4qm}}{\sqrt{n}} \gtrsim \frac{\sqrt{qm}}{\sqrt{n}}.$$

Proof technique: Charge fluctuation



Charge fluctuation $\chi \coloneqq \langle \mathfrak{c}_0 | H_S | \mathfrak{c}_0 \rangle - \langle \mathfrak{c}_1 | H_S | \mathfrak{c}_1 \rangle$.

$|\mathfrak{c}_0\rangle$ and $|\mathfrak{c}_1\rangle$: codewords corresponding to the largest and smallest eigenvalues of H_L.

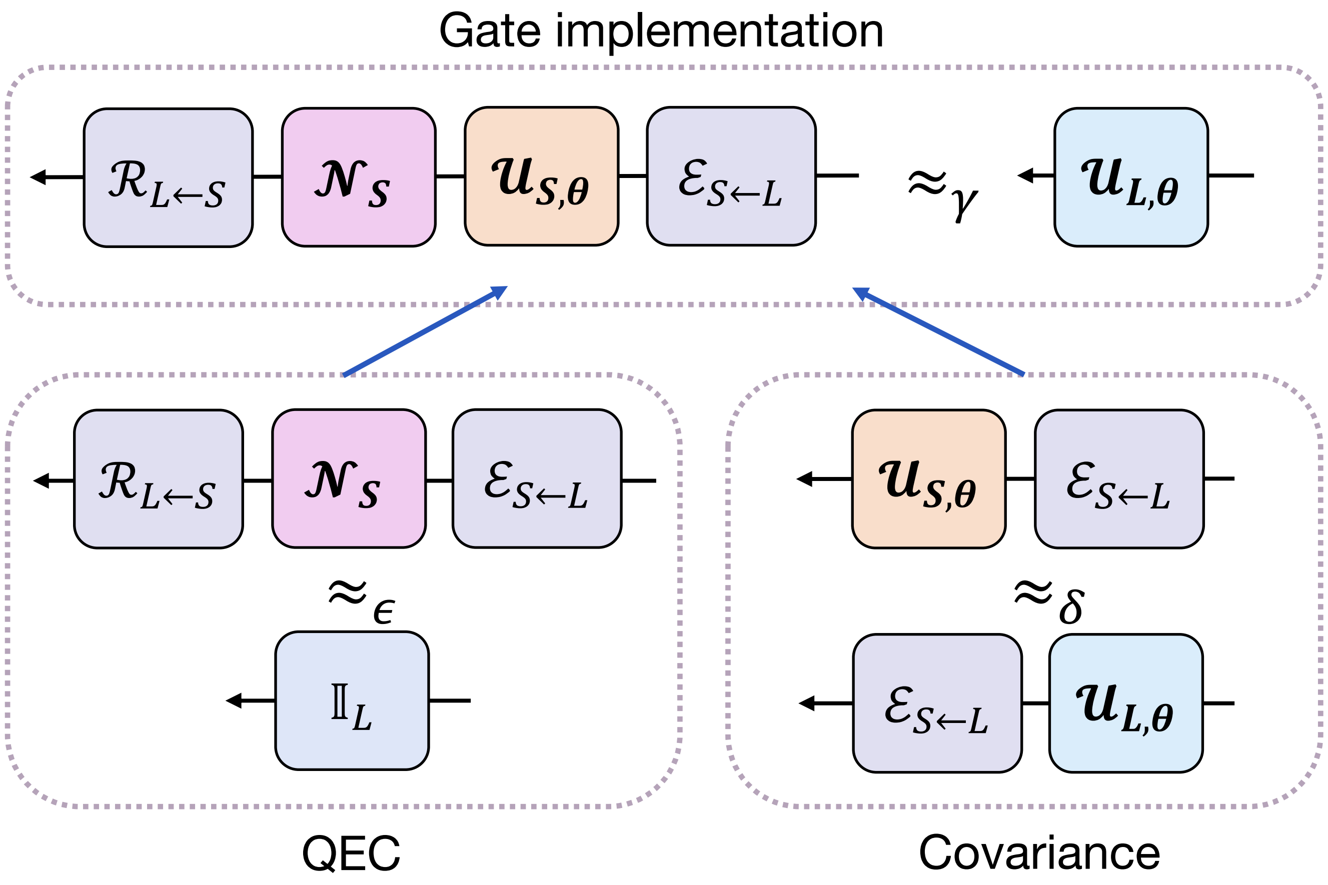
When $\epsilon = 0$, $\chi = 0$ because $P H_S P \propto P$ from HKS and QEC conditions;

[Faist *et al.* PRX'20]

When $\delta = 0$, $\chi = \Delta H_L$ because $W^\dagger H_S W = H_L - \nu \mathbb{I}_L$ (W is the encoding isometry).

$$\delta \gtrsim \sqrt{\frac{\Delta H_L - 2\epsilon\mathfrak{I}}{\Delta H_S}} \Longleftarrow \delta \gtrsim \sqrt{\frac{|\Delta H_L - \chi|}{\Delta H_S}} + |\chi| \le 2\epsilon\mathfrak{I}$$

Proof technique: Gate implementation error



$$\delta + \epsilon \gtrsim \frac{\Delta H_L}{\sqrt{4\mathfrak{F}}}$$

$\Uparrow$

$$\delta + \epsilon \geq \gamma$$

+

$$\gamma \gtrsim \frac{\Delta H_L}{\sqrt{4\mathfrak{F}}} \qquad \mathfrak{F} \gtrsim \frac{(\Delta H_L)^2}{4\gamma^2}$$

γ: Distance measure between $\mathcal{R}_{L\leftarrow S} \circ \mathcal{N}_S \circ \mathcal{U}_{S,\theta} \circ \mathcal{E}_{S\leftarrow L}$ and $\mathcal{U}_{L,\theta}$

Summary and outlook

- Tradeoff relations between QEC and continuous symmetries.
- The relations are near-optimal in certain scenarios.
- Application in fault-tolerant quantum computation.
- Other tradeoff relations e.g., based on different symmetry measures; Detailed proof techniques based on quantum metrology, quantum resource theory, etc.
- Potential physical applications in quantum gravity (AdS/CFT, black hole evaporation), many-body physics, etc.

Thank you!