Beyond i.i.d. in the Resource Theory of Asymmetry: An Information-Spectrum Approach for Quantum Fisher Information

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Joint work with
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[KY and Hiroyasu Tajima, arXiv:2204.08439]

Energetic coherence as a resource

Energetic coherence
= superposition between eigenstates of the Hamiltonian with different eigenvalues.

This resource is mandatory for

- Creating accurate clocks
- Accelerating quantum operations
- Measuring physical quantities that do not commute with the Hamiltonian

Resource manipulation in RTA

The resource theory of (time-translation) asymmetry (RTA) is a branch of resource theories that investigates the nature of energetic coherence.

In any resource theory, resource manipulation is an essential.

Distillation = an operation of extracting as much resource as possible from a given state
This is important in repairing a damaged resource.

Dilution = an operation of creating a desired state from as little resource as possible
This is important in preparing a desired state.

Several important results are known. For example,

- Exact convertibility among pure states [G. Gour and R. W. Spekkens (2008), I. Marvian PhD thesis (2012)]
- Asymptotic conversion theory for i.i.d. states $\phi^{\otimes m} \rightarrow \psi^{\otimes n}$ [I. Marvian, arXiv:2112.04694]

However, resource conversion in the non-i.i.d. regime has not been established.

Non-i.i.d. theories in entanglement theory

Non-i.i.d. theories are established e.g., in the resource theory of entanglement.

This is established with the information-spectrum method.

The information-spectrum method is a powerful method to analyze the non-i.i.d. regime for problems related to entropy.

Ent. cost: $E_{\mathbf{cost}}$ =(the minimal rate of Bell states required to crate a sequence of states)

Dist. ent.: $E_{\mathbf{dist}}$ = (the maximal rate of Bell states extractable from a sequence of states)

For **any** sequence of pure states $\hat{\psi}$, they are given by the spectral sup- and inf-entropy rates $\overline{S}$, $\underline{S}$

$$E_{\mathrm{cost}}(\hat{\psi}) = \overline{S}(\hat{\rho})$$

[G. Bowen and N. Datta (2008)]

$$E_{\mathrm{dist}}(\hat{\psi}) = \underline{S}(\hat{\rho})$$

[M. Hayashi (2003)]

Non-i.i.d. theory for RTA?

So far, it has not been possible to apply the information-spectrum method to RTA.

This is because a standard measure of energetic coherence in RTA is
the quantum Fisher information (QFI), which is quite different from entropy.

We here propose an information-spectrum approach for QFI to establish non-i.i.d. theory in RTA.

Main achievements

[KY and Hiroyasu Tajima, arXiv:2204.08439]

Main achievements are three:

1. We introduce new quantities, termed **the spectral sup- and inf-QFI rates**

   [They are the counterparts of the spectral entropy rates.]

2. To construct the spectral sup- and inf-QFI rates through the smoothing method, we define **the max- and min-QFI**

   [They are the counterparts of max- and min-entropies]

3. To show the properties of the max- and min-QFI, we introduce the notion of **asymmetric majorization** for probability distributions. We show that the exact convertibility between pure states in RTA is expressed by an asymmetric majorization relation.

   [This is the counterpart of Nielsen's theorem]

Outline of this talk

- Introduction
- Resource theory of asymmetry
- The spectral QFI rates
- Main theorem:
    - the coherence cost, the distillable coherence and the spectral QFI rates
- Intuitive explanation of the main theorem

Symmetric state and asymmetric state

Consider a quantum system with a Hamiltonian H.

Energetic coherence = superposition of energy eigenstates with different energies.

A state has energetic coherence iff it evolves by time-translation: $e^{-\mathrm{i}Ht}\rho e^{\mathrm{i}Ht} \neq \rho$, i.e., $[\rho, H] \neq 0$

Asymmetric state

A state w/o energetic coherence is invariant under time-translation : $e^{-\mathrm{i}Ht}\rho e^{\mathrm{i}Ht} = \rho$, i.e., $[\rho, H] = 0$

Symmetric state

Covariant operation

To manipulate energetic coherence, we now consider transformations of states.

A basic element is an operation that does not create energetic coherence in the sense that it transforms a symmetric state to a symmetric state.

This condition is satisfied if a CPTP map $\mathcal{E}$ describing the transformation satisfies

$$(*) \qquad \mathcal{E}\left(e^{-\mathrm{i}Ht}\rho e^{\mathrm{i}Ht}\right) = e^{-\mathrm{i}Ht}\mathcal{E}(\rho)e^{\mathrm{i}Ht} \quad \forall\rho\ ,\ \forall t \in \mathbb{R}$$

For a symmetric state ρ, $\mathcal{E}(\rho) = \mathcal{E}\left(e^{-\mathrm{i}Ht}\rho e^{\mathrm{i}Ht}\right)$

From condition $(*)$., this implies $\mathcal{E}(\rho) = e^{-\mathrm{i}Ht}\mathcal{E}(\rho)e^{\mathrm{i}Ht}$. Therefore, $\mathcal{E}(\rho)$ is symmetric.

A CPTP map satisfying condition $(*)$ is called covariant.

The resource theory of asymmetry

Any resource theory is defined by its free states that can be freely prepared and free operations that can be freely performed.

In RTA, Free state = symmetric state

Free operation = covariant operation

Any state that is not free is regarded as a resource state.

Resource state = asymmetric state

This construction is the same as in entanglement theory, where entanglement becomes a resource by defining separable states and LOCC as free states and free operations.

QFI as a standard measure

A resource measure R satisfies

1. $R(\rho) \geq R\big(\mathcal{E}(\rho)\big)$ for any free operation $\mathcal{E}$
2. $R(\rho) = 0$ for any free state ρ

A crucial resource measure in RTA is the symmetric logarithmic derivative Fisher information w.r.t. $\left\{e^{-\mathrm{i}Ht}\rho e^{\mathrm{i}Ht}\right\}_t$, given by

$$\mathcal{F}(\rho) = 2\sum_{i,j}\frac{\left(\lambda_i - \lambda_j\right)^2}{\lambda_i + \lambda_j}|\langle i|H|j\rangle|^2$$

Eigenvalue decomposition:

$$\rho = \sum_i \lambda_i |i\rangle\langle i|$$

Conventionally, this quantity is called the Quantum Fisher information (QFI).

For a pure state $\psi = |\psi\rangle\langle\psi|$, it is proportional to the energy variance: $\mathcal{F}(\psi) = 4V_H(\psi)$.

Sidenote: RTA and the resource theory of coherence

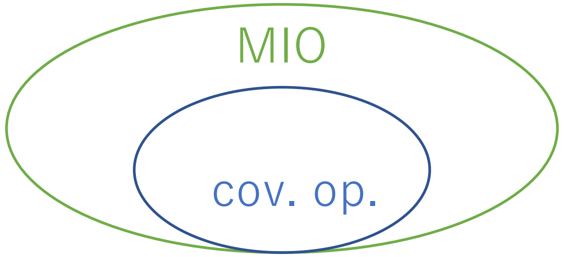
Consider a harmonic oscillator system w/ Hamiltonian $H = \sum_n n|n\rangle\langle n|$

Let us compare two different states $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |2\rangle)$

RTA w/ $H = \sum_n n|n\rangle\langle n|$

$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |2\rangle)$ are **not interconvertible**.

$\mathcal{F}(\psi_1) = 1 \leq \mathcal{F}(\psi_2) = 4$



Resource theory of coherence w.r.t. $\{|n\rangle\}_n$.

Free operation:

maximally incoherent operation= the set of all channels that map diagonal states to diagonal states

$|\psi_1\rangle$ and $|\psi_2\rangle$ are **interconvertible** since $|1\rangle \leftrightarrow |2\rangle$ is a MIO.

Hamiltonian for harmonic oscillators

From now on, we assume that the Hamiltonian is given by the harmonic oscillator Hamiltonian for simplicity.

$$H = \sum_{n=0}^{\infty} n\,|n\rangle\langle n|$$

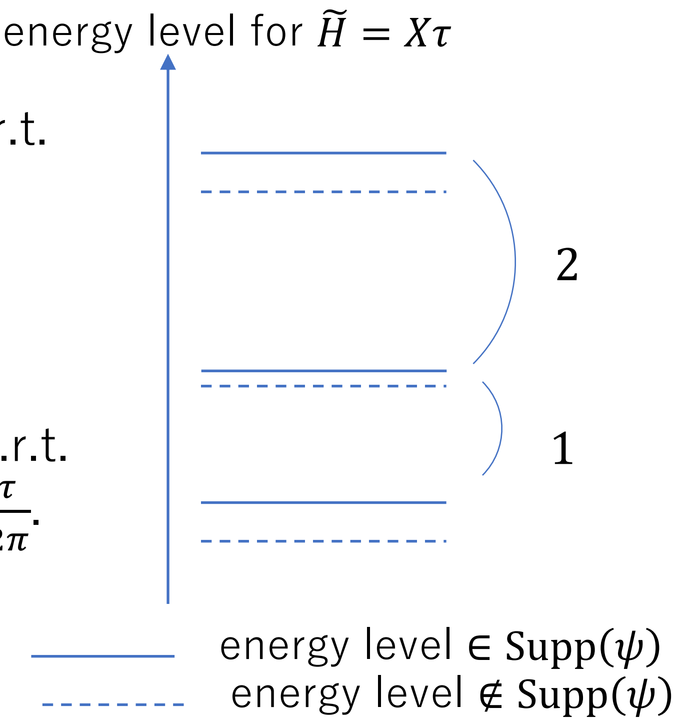
Conversion theory for harmonic oscillators in pure states can be generalized to any systems in periodic pure states with an arbitrary Hamiltonian.

[I. Marvian, arXiv:2112.04694]

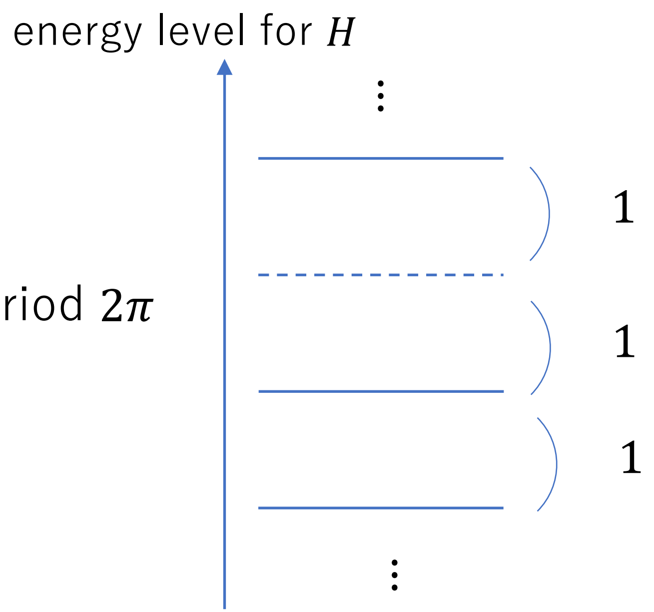
[KY and Hiroyasu Tajima, arXiv:2204.08439]

$|\psi\rangle$ has a period τ w.r.t. a Hamiltonian X.

$|\psi\rangle$ has a period 2π w.r.t. a Hamiltonian $\widetilde{H} = X\frac{\tau}{2\pi}$.



$|\psi\rangle$ has a period 2π w.r.t H



Asymptotic convertibility

We adopt the trace distance $D(\rho,\sigma) \coloneqq \frac{1}{2}\|\rho-\sigma\|_1$ as a quantifier of error.

We say that a sequence $\hat{\rho}=\{\rho_m\}_m$ is convertible to another sequence $\hat{\sigma}=\{\sigma_m\}_m$ by cov. op. iff

$$\exists \text{ covariant operations } \{\mathcal{E}_m\}_m \text{ s.t. } \lim_{m\to\infty} D(\mathcal{E}_m(\rho_m),\sigma_m)=0.$$

In this case, we denote $\hat{\rho} \overset{\text{cov}}{\succ} \hat{\sigma}$.

We introduce two key quantities: the coherence cost and the distillable coherence

$$C_{\text{cost}}(\hat{\rho}) \coloneqq \inf\left\{R \mid \widehat{\phi_{\text{coh}}}(R) \overset{\text{cov}}{\succ} \hat{\rho}\right\} \qquad C_{\text{dist}}(\hat{\rho}) \coloneqq \sup\left\{R \mid \hat{\rho} \overset{\text{cov}}{\succ} \widehat{\phi_{\text{coh}}}(R)\right\}$$

Coherence bit: $|\phi_{\text{coh}}\rangle \coloneqq \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$, $\phi_{\text{coh}} \coloneqq |\phi_{\text{coh}}\rangle\langle\phi_{\text{coh}}|$, $\widehat{\phi_{\text{coh}}}(R) \coloneqq \left\{\phi_{\text{coh}}^{\otimes\lceil Rm\rceil}\right\}_m$

In the i.i.d. regime, $C_{\text{cost}}(\hat{\psi}) = C_{\text{dist}}(\hat{\psi}) = \mathcal{F}(\psi)$ holds for $\hat{\psi}=\{\psi^{\otimes m}\}_m$ with a pure state ψ with period 2π.

[I. Marvian, arXiv:2112.04694]

Notations

Energy distribution:

For a given pure state ψ, we denote $p_\psi(n) \coloneqq |\langle n|\psi\rangle|^2$.
Energy distribution plays a key role since $|\psi\rangle$ is reversibly convertible to $\sum_n \sqrt{p(n)}|n\rangle$

Product (convolution) $*$:

For two sequences of numbers $a = \{a(n)\}_n$ and $b = \{b(n)\}_n$, we define their product $a * b$ by

$$a * b(n) \coloneqq \sum_{k\in\mathbb{Z}} a(k)b(n-k)$$

Inverse sequence $\tilde{q}$:

For a given sequence q, we say another sequence $\tilde{q}$ satisfies $\tilde{q} * q(n) = \delta_{0,n}$.

If there exists $n_* = \min\{n|q(n) > 0\}$, then the unique $\tilde{q}$ can be explicitly constructed by a recursive formula.

Generalized Poisson distribution

Generalized Poisson distribution:

For $\lambda \in \mathbb{R}$, we define $\mathrm{P}_\lambda = \{\mathrm{P}_\lambda(n)\}$ by $\mathrm{P}_\lambda(n) := \begin{cases} e^{-\lambda}\dfrac{\lambda^n}{n!} & (n \geq 0) \\ 0 & (n < 0) \end{cases}$

For $\lambda \geq 0$, it is an ordinary Poisson distribution.
For $\lambda < 0$, it is **not** a probability distribution. Nevertheless, it plays an important role since $\widetilde{\mathrm{P}_\lambda} = \mathrm{P}_{-\lambda}$

The max- and min-QFI

We introduce two key quantities for a pure state ψ:

$$\mathcal{F}_{\max}(\psi) \coloneqq \inf\left\{4\lambda | \mathrm{P}_{\lambda} * \widetilde{p_{\psi}} \geq 0\right\} \qquad \mathcal{F}_{\min}(\psi) \coloneqq \sup\left\{4\lambda | p_{\psi} * \mathrm{P}_{-\lambda} \geq 0\right\}$$

The max- and min-QFI are the amounts of energetic coherence in ψ that can be transformed from and to a pure state whose energy distribution is given by the Poisson distribution.

The max-QFI is also defined for a general state ρ by $\mathcal{F}_{\max}(\rho) \coloneqq \inf_{\Phi_{\rho}, H_A} \mathcal{F}_{\max}(\Phi_{\rho})$

(Φ_{ρ}: purification of ρ, H_A: the Hamiltonian of ancilla)

The max- and min-QFIs have similar properties to the max- and min-entropies. For example,

$$\mathcal{F}_{\max}(\psi) \geq \mathcal{F}(\psi) \geq \mathcal{F}_{\min}(\psi)$$

[KY and Hiroyasu Tajima, arXiv:2204.08439]

The spectral QFI rates

We define the spectral sup-and inf-QFI rates by

$$\overline{\mathcal{F}}(\hat{\psi}) \coloneqq \lim_{\epsilon\to 0} \limsup_{m\to\infty} \frac{1}{m} \mathcal{F}^{\epsilon}_{\max}(\psi_m) \qquad \underline{\mathcal{F}}(\hat{\psi}) \coloneqq \lim_{\epsilon\to 0} \liminf_{m\to\infty} \frac{1}{m} \mathcal{F}^{\epsilon}_{\min}(\psi_m)$$

where the smooth max- and min-QFIs are defined by

$$\mathcal{F}^{\epsilon}_{\max}(\psi) \coloneqq \inf_{\rho\in B^{\epsilon}(\psi)} \mathcal{F}_{\max}(\rho) \qquad \mathcal{F}^{\epsilon}_{\min}(\psi) \coloneqq \sup_{\rho\in B^{\epsilon}_{\mathrm{pure}}(\psi)} \mathcal{F}_{\min}(\rho)$$

$$B^{\epsilon}(\rho) \coloneqq \{\text{states } \rho \,|\, D(\rho,\sigma) \le \epsilon\} \qquad B^{\epsilon}_{\mathrm{pure}}(\rho) \coloneqq \{\text{pure states } \phi \,|\, D(\rho,\phi) \le \epsilon\}$$

[KY and Hiroyasu Tajima, arXiv:2204.08439]

Cf. The spectral entropy rates w/ smoothing method:

$$\overline{S}(\hat{\rho}) \coloneqq \lim_{\epsilon\to 0} \limsup_{m\to\infty} \frac{1}{m} S^{\epsilon}_{\max}(\psi_m) \qquad \underline{S}(\hat{\rho}) \coloneqq \lim_{\epsilon\to 0} \liminf_{m\to\infty} \frac{1}{m} S^{\epsilon}_{\min}(\psi_m)$$

$$S^{\epsilon}_{\max}(\psi) \coloneqq \inf_{\rho\in B^{\epsilon}(\psi)} S_{\max}(\rho) \qquad S^{\epsilon}_{\min}(\psi) \coloneqq \sup_{\rho\in B^{\epsilon}(\psi)} S_{\min}(\rho)$$

[N. Datta and R. Renner (2009)]
[R. Renner, PhD thesis (2005)]

Main theorem

Main result [KY and Hiroyasu Tajima, arXiv:2204.08439]

For **any** sequence of pure states $\hat{\psi} = \{\psi_m\}_m$,

$$C_{\mathrm{cost}}(\hat{\psi}) = \overline{\mathcal{F}}(\hat{\psi}) \qquad C_{\mathrm{dist}}(\hat{\psi}) = \underline{\mathcal{F}}(\hat{\psi})$$

The spectral QFI rates are defined by $\overline{\mathcal{F}}(\hat{\psi}) := \lim_{\epsilon\to 0}\limsup_{m\to\infty}\frac{1}{m}\mathcal{F}_{\max}^{\epsilon}(\psi_m)$, $\underline{\mathcal{F}}(\hat{\psi}) := \lim_{\epsilon\to 0}\liminf_{m\to\infty}\frac{1}{m}\mathcal{F}_{\min}^{\epsilon}(\psi_m)$

In entanglement theory

For **any** sequence of pure states $\hat{\psi} = \{\psi_m\}_m$,

$$E_{\mathrm{cost}}(\hat{\psi}) = \overline{S}(\hat{\rho}) \qquad E_{\mathrm{dist}}(\hat{\psi}) = \underline{S}(\hat{\rho}) \qquad (\hat{\rho} = \{\rho_m\},\quad \rho_m = \mathrm{Tr}_B(\psi_{AB,m}))$$

[M. Hayashi (2003), G. Bowen and N. Datta (2008)]

The spectral entropy rates are given by $\overline{S}(\hat{\rho}) := \lim_{\epsilon\to 0}\limsup_{m\to\infty}\frac{1}{m}S_{\max}^{\epsilon}(\rho_m)$, $\underline{S}(\hat{\rho}) := \lim_{\epsilon\to 0}\liminf_{m\to\infty}\frac{1}{m}S_{\min}^{\epsilon}(\rho_m)$

[N. Datta and R. Renner (2009)]

a-majorization

We here introduce a notion of asymmetric-majorization (a-majorization).

For given two probability distributions $p = \{p(n)\}_n$ and $q = \{q(n)\}_n$, we say that p a-majorizes q iff

$$p * \tilde{q}(n) \geq 0 \text{ for all } n \in \mathbb{Z}$$

In this case, we denote $p \succ_a q$.

A key result:

A pure state ψ is convertible to ϕ by a covariant operation w/o error iff $p_\psi \succ_a p_\phi$

[KY and Hiroyasu Tajima, arXiv:2204.08439]

[Other forms of NS condition for the exact conversion can be found e.g., in G. Gour and R. W. Spekkens (2008)]

(cf.) Nielsen's theorem in entanglement theory:

A pure state ψ is convertible to ϕ by a LOCC w/o error iff $\lambda_\psi \succ \lambda_\phi$

λ_ψ: the prob. distr. defined by the Schmidt coefficients of a bipartite pure state ψ_{AB}

Interpretation of the max- and min-QFIs

The max-QFI is defined by

$$\mathcal{F}_{\max}(\psi) \coloneqq \inf\left\{4\lambda \middle| \mathrm{P}_\lambda * \widetilde{p_\psi} \geq 0\right\}$$

$$= \inf\{4\lambda|\ |\chi_\lambda\rangle \text{ is convertible to } |\psi\rangle\}$$

$$|\chi_\lambda\rangle \coloneqq \sum_n \sqrt{\mathrm{P}_\lambda(n)}|n\rangle$$

$$= \inf\{\mathcal{F}(\chi_\lambda)|\ |\chi_\lambda\rangle \text{ is convertible to } |\psi\rangle\}$$

The minimal amount of energetic coherence (i.e., QFI) in $|\chi_\lambda\rangle$ that is required to create $|\psi\rangle$.

The min-QFI is defined by

$$\mathcal{F}_{\min}(\psi) \coloneqq \sup\left\{4\lambda \middle| p_\psi * \mathrm{P}_{-\lambda} \geq 0\right\}$$

$$= \sup\{4\lambda|\ |\psi\rangle \text{ is convertible to } |\chi_\lambda\rangle\}$$

$$|\chi_\lambda\rangle \coloneqq \sum_n \sqrt{\mathrm{P}_\lambda(n)}|n\rangle$$

$$= \sup\{\mathcal{F}(\chi_\lambda)|\ |\psi\rangle \text{ is convertible to } |\chi_\lambda\rangle\}$$

The maximum amount of energetic coherence (i.e., QFI) in $|\chi_\lambda\rangle$ that can be extracted from $|\psi\rangle$.

These observations are essential to show

$$C_{\mathrm{cost}}(\hat{\psi}) = \overline{\mathcal{F}}(\hat{\psi}) = \lim_{\epsilon\to 0}\limsup_{m\to\infty}\frac{1}{m}\mathcal{F}_{\max}^{\epsilon}(\psi_m) \qquad C_{\mathrm{dist}}(\hat{\psi}) = \underline{\mathcal{F}}(\hat{\psi}) = \lim_{\epsilon\to 0}\liminf_{m\to\infty}\frac{1}{m}\mathcal{F}_{\min}^{\epsilon}(\psi_m)$$

See [KY and Hiroyasu Tajima, arXiv:2204.08439] for technical details.

Cost for mixed states

Let us find a minimal cost required to create a mixed state ρ from $|\chi_\lambda\rangle = \sum_{n=0}^{\infty} \sqrt{\mathrm{P}_\lambda(n)}\,|n\rangle$.

Since the partial trace is a covariant operation, (Cost of Φ_ρ) $\geq$ (Cost of ρ) for any purification Φ_ρ.

By using the covariant Stinespring dilation theorem, we find $\inf_{\Phi_\rho}$(Cost of Φ_ρ) $\leq$ (Cost of ρ).

cov. Stinespring dilation:

For any covariant operation $\mathcal{E}$, $\exists$ an ancilla A with Hamiltonian H_A, its eigenstate $|\eta_A\rangle$ and an energy-preserving (covariant) unitary U_{SA} s.t. $\mathcal{E}(\sigma_S) = Tr_A\left(U_{SA}(\sigma_S \otimes |\eta_A\rangle\langle\eta_A|)U_{SA}^\dagger\right)$

If ρ is created from $|\chi_\lambda\rangle$ by $\mathcal{E}$, a purification $\Phi_\rho \coloneqq U_{SA}(|\chi_\lambda\rangle\langle\chi_\lambda| \otimes |\eta_A\rangle\langle\eta_A|)U_{SA}^\dagger$ can also be created.

Therefore, $\inf_{\Phi_\rho}$(Cost of Φ_ρ) = (Cost of ρ).

- The max-QFI is defined as $\mathcal{F}_{\max}(\rho) \coloneqq \inf_{\Phi_\rho, H_A} \mathcal{F}_{\max}(\Phi_\rho)$
- Furthermore, it is shown $C_{\mathrm{cost}}(\hat{\rho}) = \overline{\mathcal{F}}(\hat{\rho})$ for **any** sequence $\hat{\rho} = \{\rho_m\}_m$, where $\overline{\mathcal{F}}(\hat{\rho}) \coloneqq \lim_{\epsilon\to 0} \limsup_{m\to\infty} \frac{1}{m} \mathcal{F}_{\max}^{\epsilon}(\rho_m)$

See [KY and Hiroyasu Tajima, arXiv:2204.08439] for furhter details.

Summary

We established conversion theory between sequences of pure states in the non-i.i.d. regime by constructing the spectral sup- and inf-QFI rates.

$$C_{\mathrm{cost}}(\hat{\psi}) = \overline{\mathcal{F}}(\hat{\psi}) \qquad \overline{\mathcal{F}}(\hat{\psi}) := \lim_{\epsilon\to 0}\limsup_{m\to\infty}\frac{1}{m}\mathcal{F}^{\epsilon}_{\max}(\psi_m)$$

$$C_{\mathrm{dist}}(\hat{\psi}) = \underline{\mathcal{F}}(\hat{\psi}) \qquad \underline{\mathcal{F}}(\hat{\psi}) := \lim_{\epsilon\to 0}\liminf_{m\to\infty}\frac{1}{m}\mathcal{F}^{\epsilon}_{\min}(\psi_m)$$

The result for cost can be directly extended to a general sequence of states.

To construct the spectral sup- and inf-QFI rates through the smoothing method, we define the max- and min-QFI.

Asymmetric majorization relation gives a necessary and sufficient condition for exact convertibility among pure states, which is the counterpart in RTA to Nielsen’s theorem.