Transcendental properties of entropy-constrained sets

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joint work with Michael M. Wolf

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(Semi)Algebraic vs Transcendental?

Functions and Sets

(semi) algebraic

$$f(x) = y \Leftrightarrow poly(x, f(x)) = 0$$

transcendental

$$\exp(x), \ln(x)$$

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Nash manifolds
Nash maps

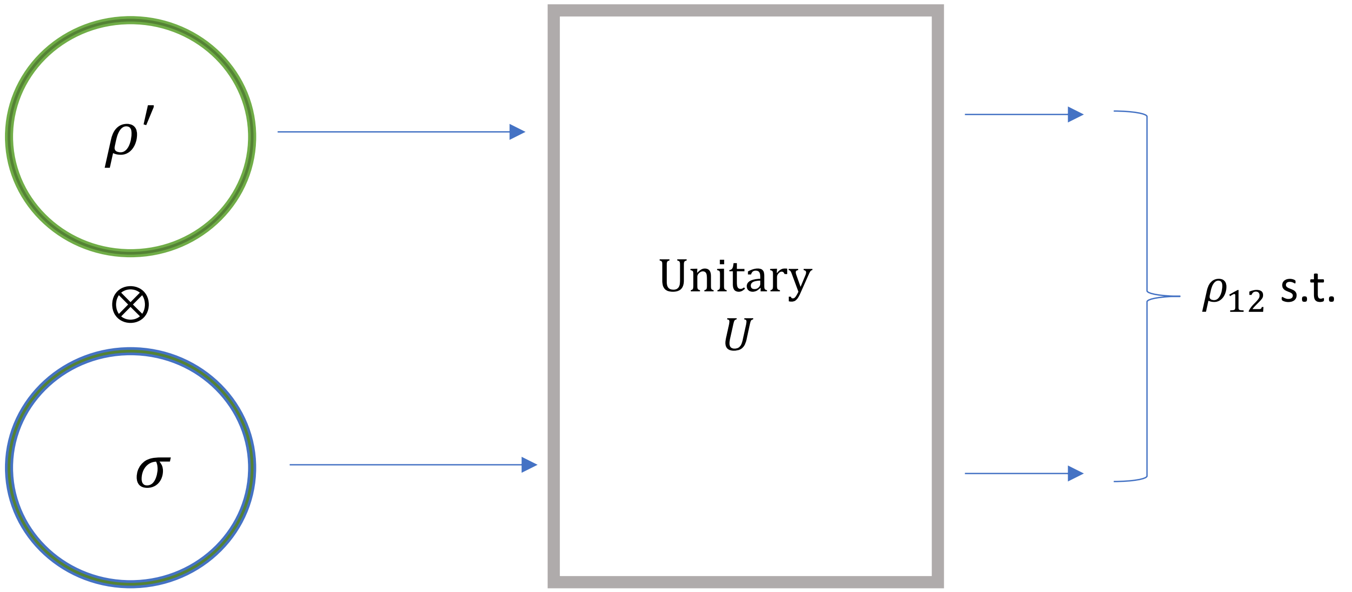
single-shot / finite

transcendental

$$\exp(x), \ln(x)$$

asymptotic / infinite

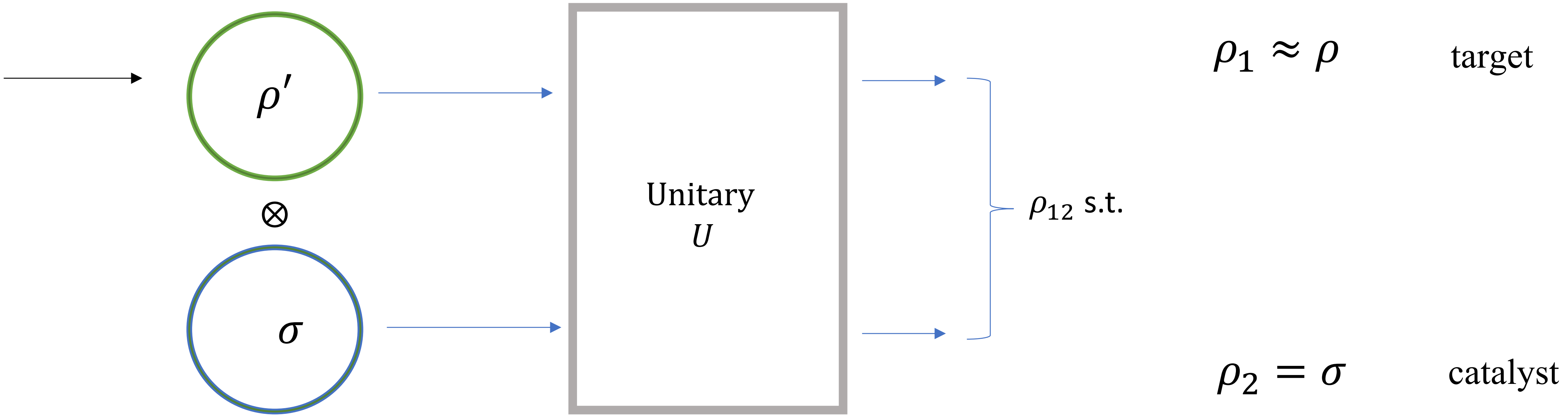
Catalytic state transformations [Wilming et al, 2019]



$\rho_1 \approx \rho$ target

$\rho_2 = \sigma$ catalyst

Catalytic state transformations [Wilming et al, 2019]



$$\mathcal{S}_n := \{\rho \in \mathbb{C}^d \; s.t. \; \forall \varepsilon > 0 \; \exists \sigma \in \mathbb{C}^n \; and \, a \; unitary \; U \in \mathbb{C}^d \otimes \mathbb{C}^n \; s.t. \; the \; reduced \; states \; of \; \rho_{12} := U(\rho' \otimes \sigma)U^* \; satisfy \; \rho_2 = \sigma \; and \; ||\rho_1 - \rho||_1 \leq \varepsilon\}$$

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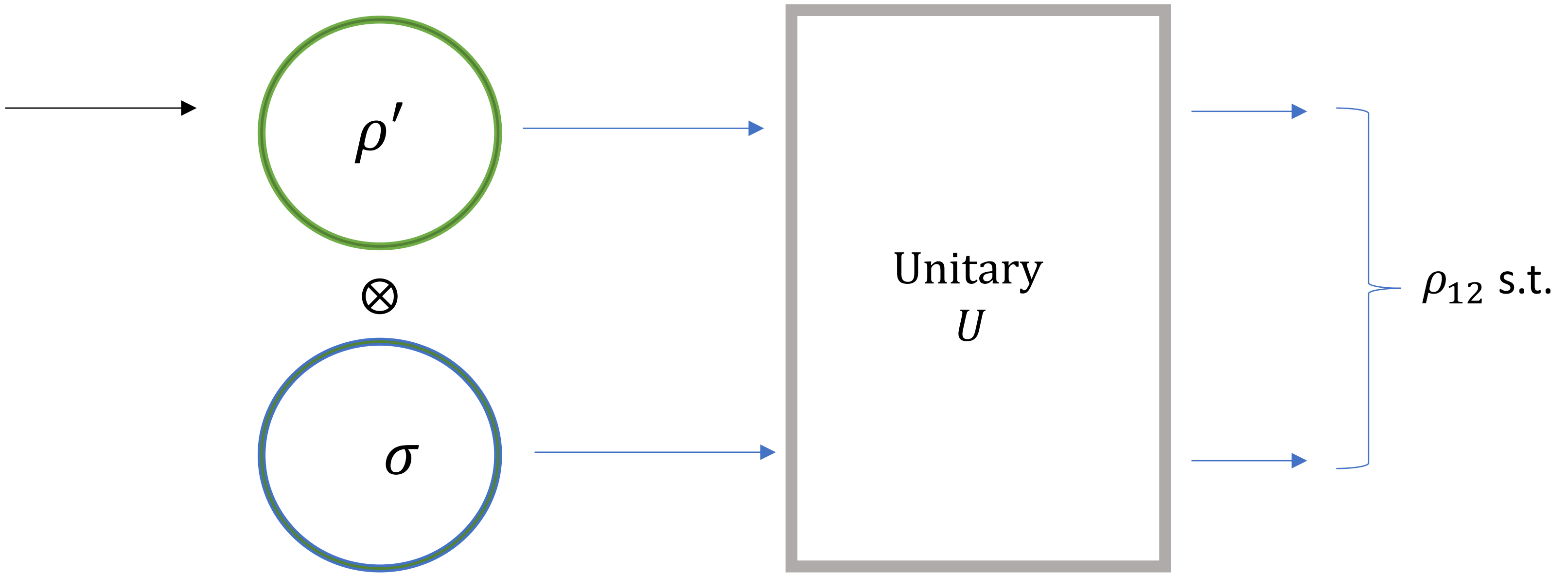

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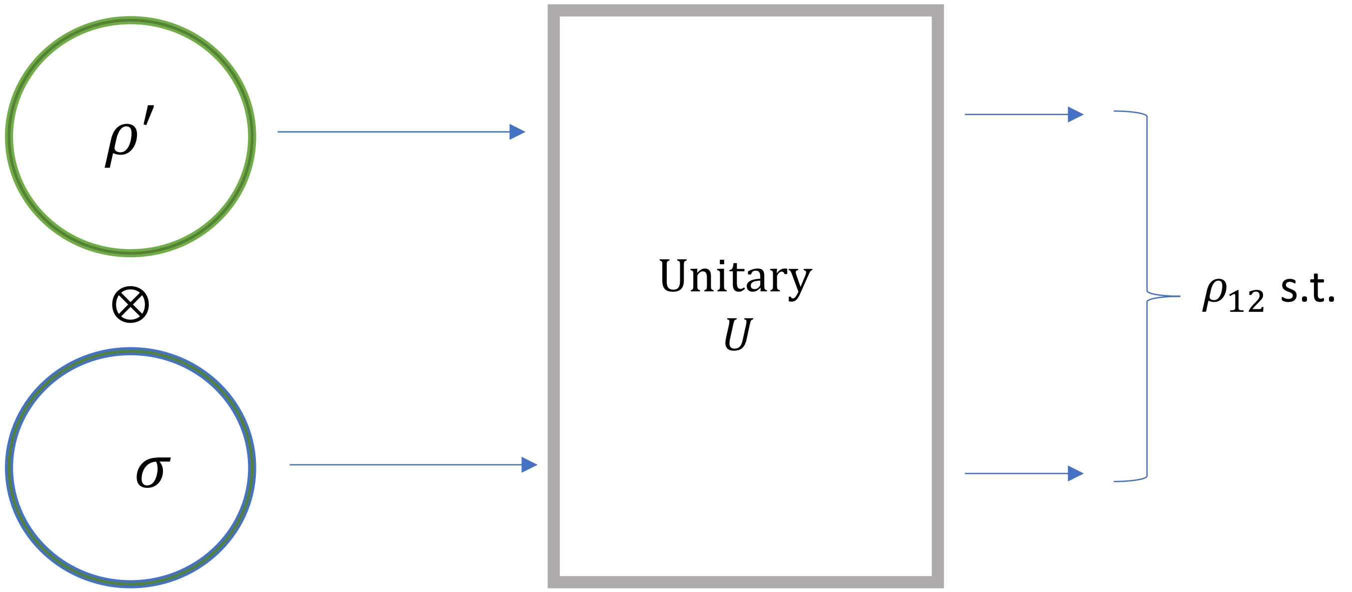

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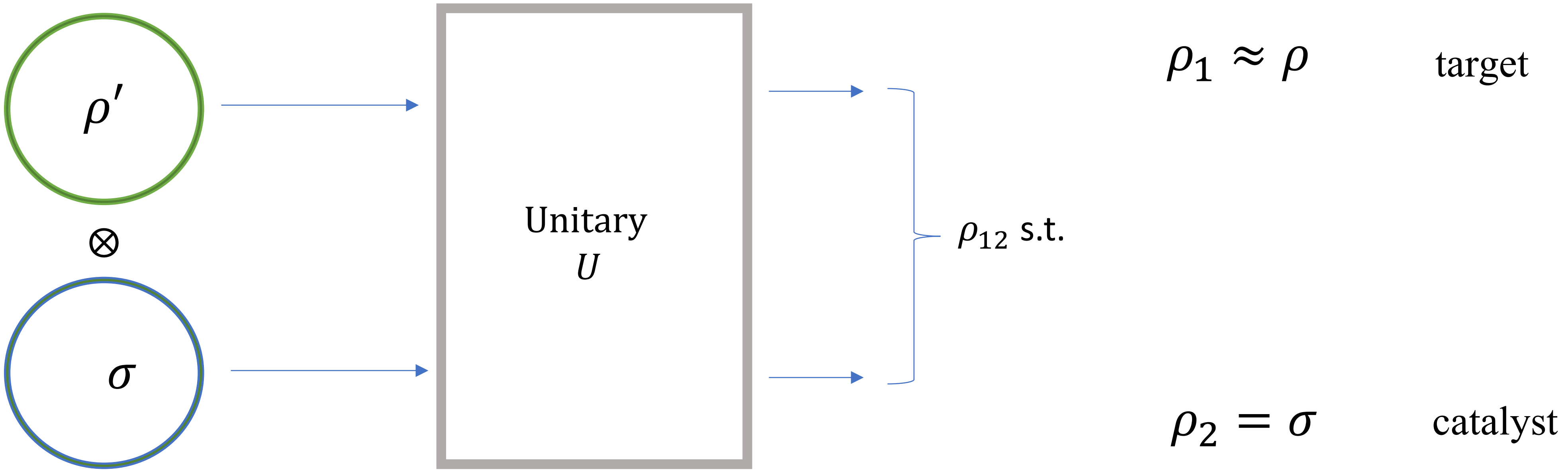
$$\lim_{n\to\infty} S_n = S \text{ but is there a finite } n = n(d) \text{ such that } S_n = S?$$

Catalytic state transformations



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semiaglebraic (courtesy of Tarski-Seidenberg)

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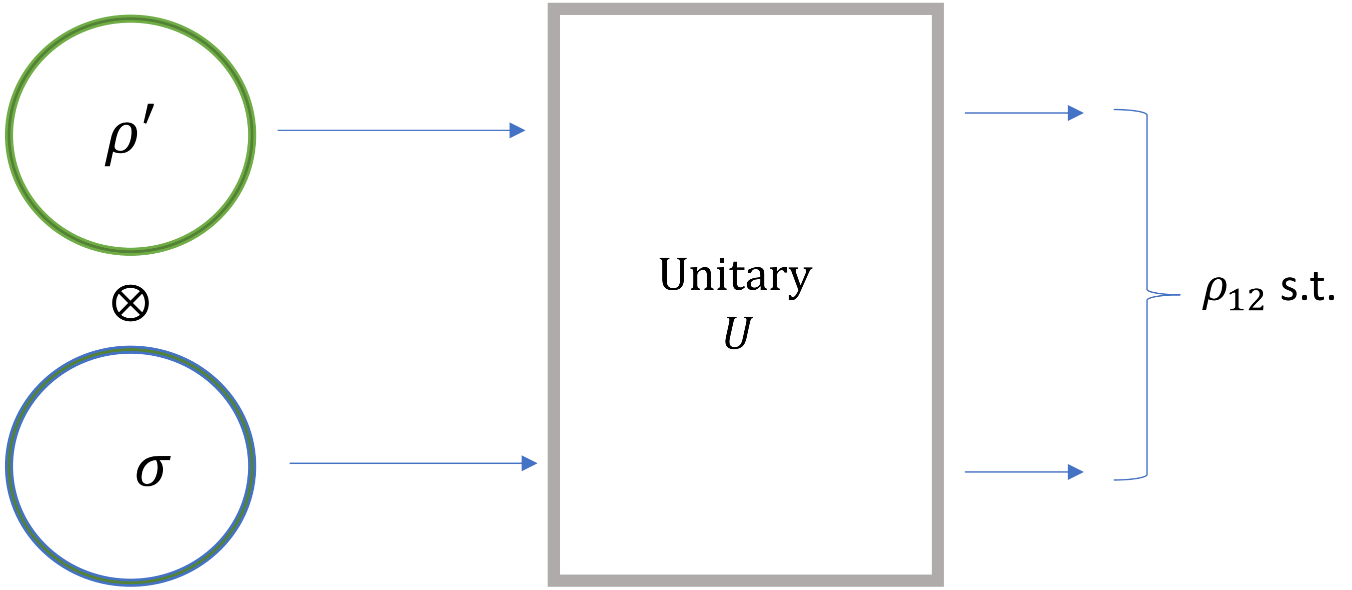

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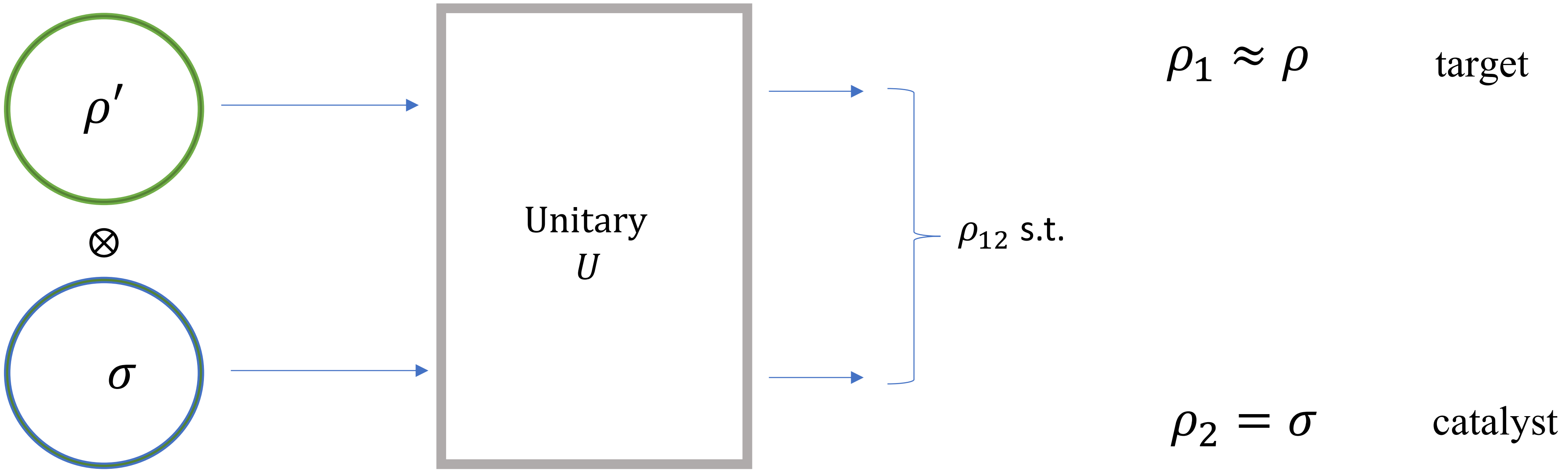

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transcendental? $\Rightarrow S_n \neq S$

Are entropy-constrained sets transcendental?

Main result

A general criteria that discriminates semialgebraic sets from non-semiaglebraic ones:

Theorem 1

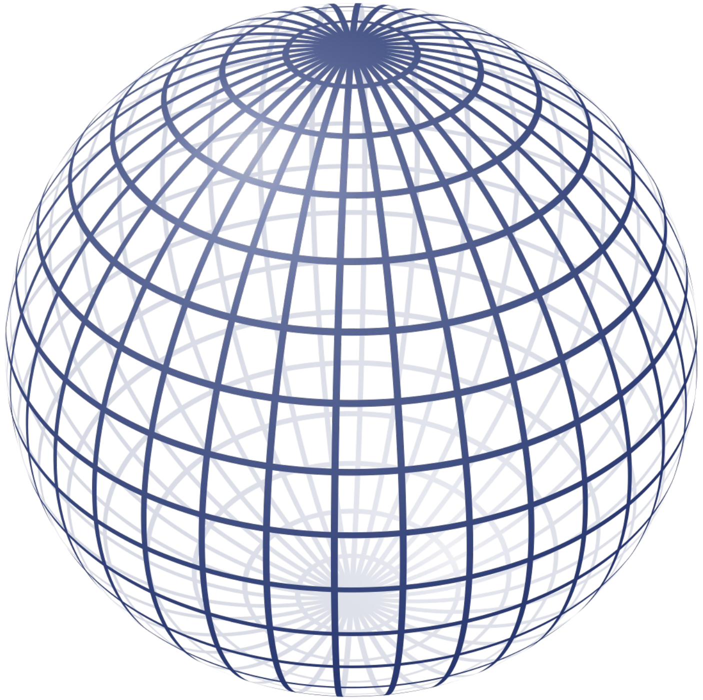
Let M be a Nash submanifold of $\mathbb{R}^n$, and $P(x) \in \mathbb{R}^{n\times n}$ the orthogonal projector onto the normal space of M at $x \in M$. For any pair of Nash maps $g : I \subseteq \mathbb{R} \to M$, I an open interval, and $h : \mathbb{R}^{n\times n} \to \mathbb{R}$ define $f := h \circ P \circ g : I \to \mathbb{R}$. Then

1. *f is analytic and algebraic over $\mathbb{R}$,*
2. *the global analytic function obtained from f by analytic continuation has a compact Riemann surface and, in particular, a finite number of branches.*

Are entropy-constrained sets transcendental?

For $d = 2$:

$$S = \{\rho \,|\, S(\rho) = c\} = \{\rho \,|\, \mathrm{tr}\left[\rho^2\right] = \tilde{c}\}$$



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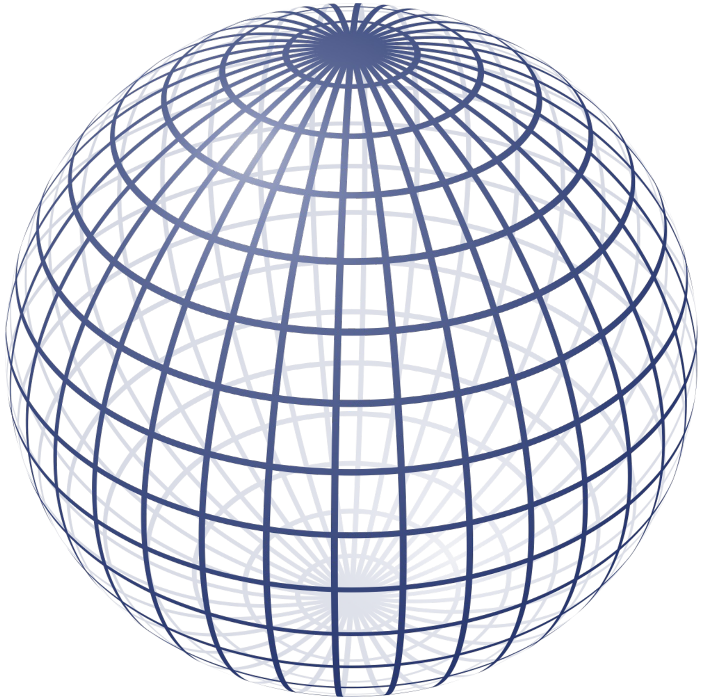
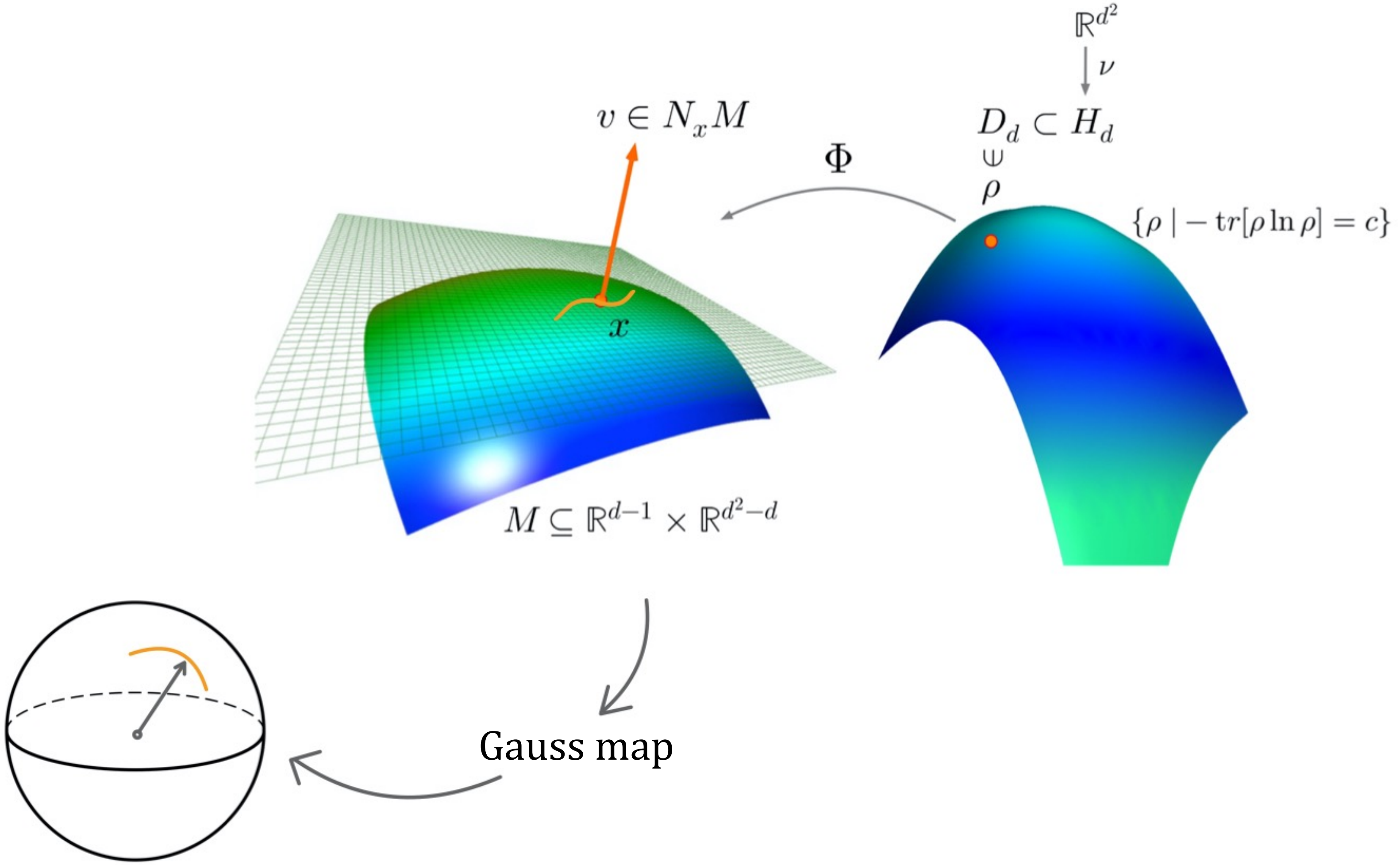


$\Longrightarrow$ Entropy-constrained sets are semialgebraic

Yes, for $d \geq 3$

Proof idea:



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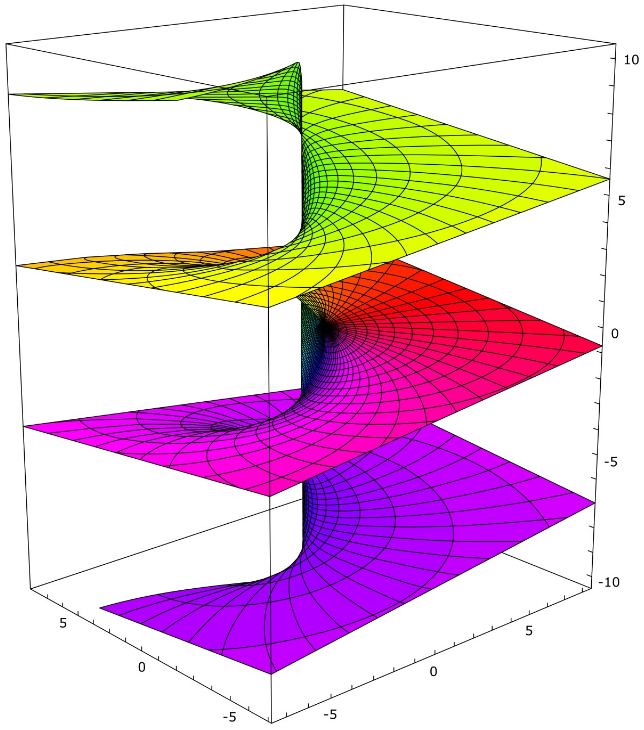

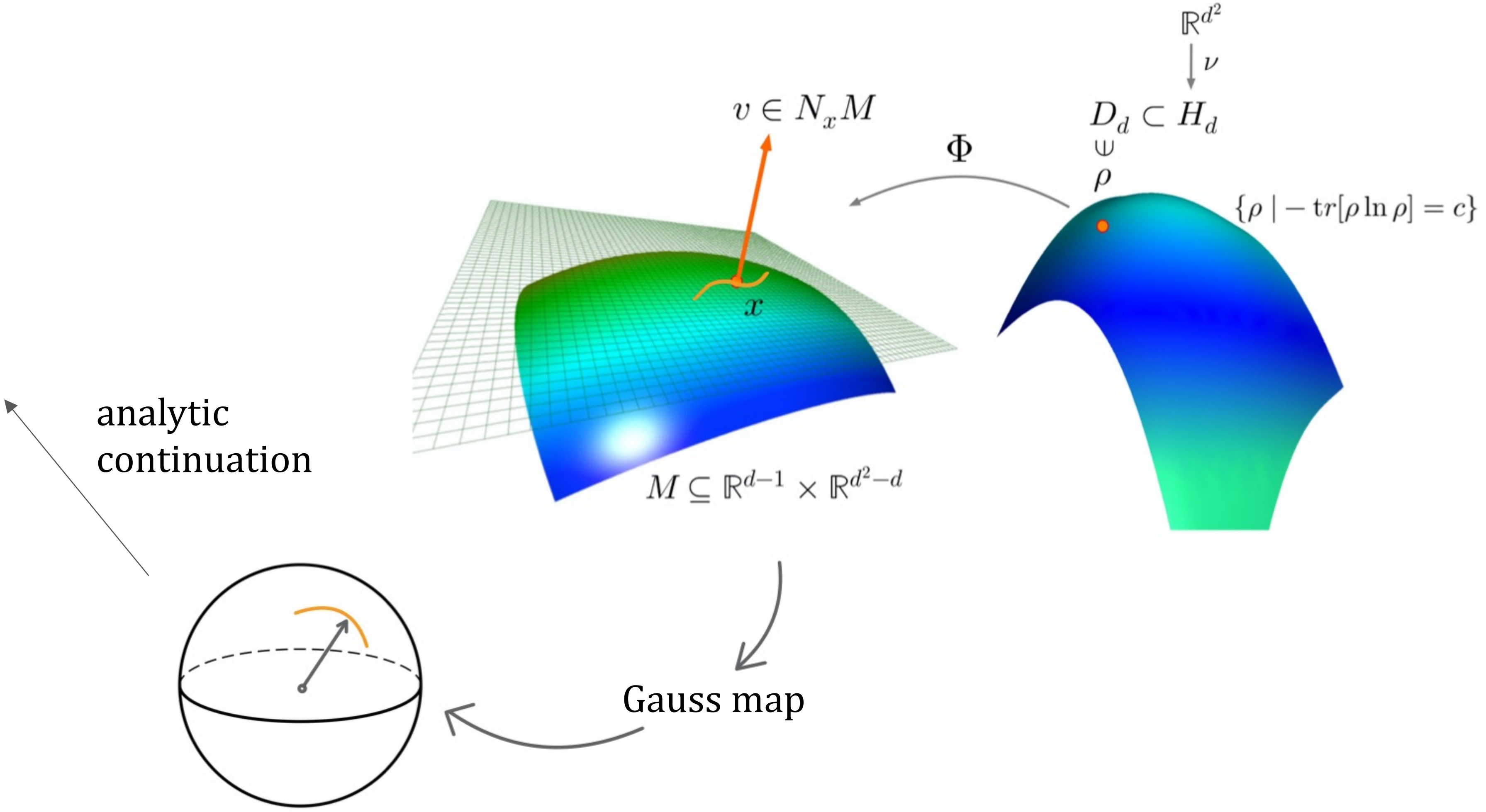

non compact
Riemann surface

Theorem 2: For any $d \geq 3$ and $c \in (0, \ln d)$ the set of $d \times d$ density operators whose von Neumann entropy is equal to c is nowhere semialgebraic. That is, if $\mathcal{S} := \{\rho \in H_d | \rho \geq 0, \mathrm{Tr}[\rho] = 1, -\mathrm{Tr}[\rho \ln \rho] = c\}$, then for any open set $V \subset H_d$ the set $\mathcal{S} \cap V$ is not semialgebraic unless it is empty.

$H_d \subseteq \mathbb{C}^{d \times d}$ denotes the space of Hermitian $d \times d$ matrices.

Remarks:

1. The same statement holds true for any other logarithmic base

2. The same result holds true for the inequalities "$< c$", "$\leq c$", "$> c$" and "$\geq c$"

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- the distance of the state to a specific target
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mathematically described using a differentiable function $h : P_d \to \mathbb{R}$

Theorem 3: For $d \geq 3$, $c_1 \in (0, \ln d)$, $c_2 \in \mathbb{R}$ and $h \in C^1(P_d,)$ let ρ be an element of

$$\mathcal{S} := \{\rho \in P_d \mid \operatorname{Tr}[\rho] = 1, -\operatorname{Tr}[\rho \ln \rho] = c_1, h(\rho) = c_2\},$$

with $[\rho, \nabla h(\rho)] \neq 0$. Then $\mathcal{S}$ is not semialgebraic in any neighborhood of ρ.

Application of Theorem 3: Transcendentality of the level sets of the relative entropy

Corollary 1: For any $c > 0$, $d \geq 3$, any positive definite density matrix $\sigma \in H_d$ and any open subset $U \subseteq H_d$ the set

$$\mathcal{R} := \left\{\rho \in D_d \mid S(\rho\|\sigma) = c\right\} \cap U$$

is not semialgebraic in H_d unless it is empty.

$$S(\rho\|\sigma) := \operatorname{tr}\left[\rho \ln \rho\right] - \operatorname{tr}\left[\rho \ln \sigma\right]$$

Take away

- $\mathcal{S} \neq \mathcal{S}_n$
- Distinction between single-shot/finite and asymptotic/infinite often comes hand-in-hand with the separation (semi)algebraic vs transcendental
- Useful for no-go results (e.g. single-shot characterizations of information theoretic quantities)

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➢ variants of the presented arguments to also apply to many other entropic quantities that appear in classical and quantum information theory

Thank you!