# Geometrical Aspects of Data-Processing of Markov Chains

Beyond IID in Information Theory 10

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# Introduction & Preliminaries

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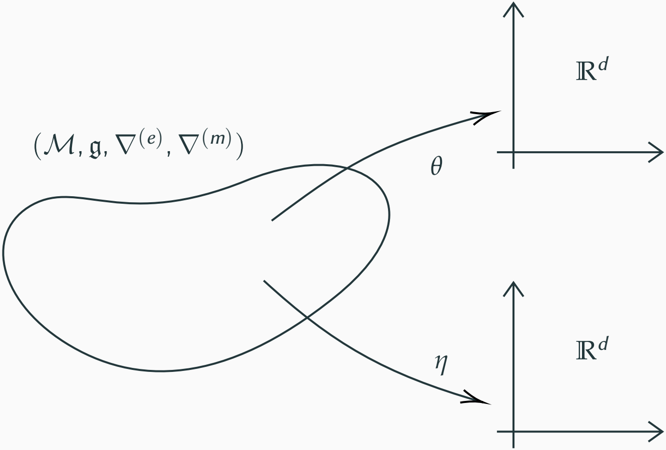

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**Fisher information metric $\mathfrak{g}$ and dual affine connections $\nabla^{(e)}, \nabla^{(m)}$**

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2. Information decomposition / projection.
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**What about Markov models?**

# Irreducible Markov chains

**Notation**

$\mathcal{E} \subset \mathcal{Y}^2$ such that $(\mathcal{Y}, \mathcal{E})$ strongly connected.

Positive functions over $\mathcal{E}$: $\mathcal{F}_+(\mathcal{Y}, \mathcal{E})$.

Irreducible row-stochastic matrices over $(\mathcal{Y}, \mathcal{E})$: $\mathcal{W}(\mathcal{Y}, \mathcal{E})$.

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$$\mathbb{P}\left(Y_1 = y_1, \dots, Y_k = y_k\right) = \mu(y_1) \prod_{t=1}^{k-1} P(y_t, y_{t+1}),$$

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Stationary distribution: $\pi P = \pi$.

Edge-measure: $Q(y, y') = \pi(y) P(y, y') = \mathbb{P}_\pi\left(Y_t = y, Y_{t+1} = y'\right)$.

# Exponential tilting (ET)

**ET of distribution**
$Y \sim \mu \in \mathcal{P}([m]), f \colon \mathcal{Y} \to \mathbb{R}$. Construct exponential family:

$$\mu_\theta(y) = \mu(y) e^{\theta f(y) - \kappa(\theta)}, \qquad \kappa(\theta) = \log \mathbb{E} e^{\theta f(Y)} \qquad \text{(CGF)}.$$

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$$\mathfrak{s}\colon \mathcal{F}_+(\mathcal{Y}, \mathcal{E}) \to \mathcal{W}(\mathcal{Y}, \mathcal{E}),\ \widetilde{P}(y, y') \mapsto P(y, y') = \frac{\widetilde{P}(y, y') v(y')}{\rho v(y)},$$

where $\rho$, $v$ are the PF root and right PF eigenvector of $\widetilde{P}$.

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# Information Geometry of Markov Chains

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1. Large deviations
   Miller (1961); Donsker and Varadhan (1975); Gärtner (1977).
2. Information projection
   Csiszár, Cover, and Choi (1987).
3. Asymptotic e-families
   Ito and Amari (1988); Takeuchi and Barron (1998); Takeuchi and Kawabata (2007); Takeuchi and Nagaoka (2017).
4. One-parameters exponential families
   Nakagawa and Kanaya (1993).
5. Dually flat structure
   Nagaoka (2005).

# Information Geometry of Markov Chains (Nagaoka, 2005)

Distributions

$\mathcal{P}(\mathcal{Y})$

$D$

KL divergence

$$D\left(\mu_\theta \| \mu_{\theta'}\right) = \mathbb{E}_{\mu_\theta} \log \frac{\mu_\theta(y)}{\mu_\theta(y')}$$

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Markov chains

$\mathcal{W}(\mathcal{Y}, \mathcal{E})$

$$Y_1, Y_2, \ldots, Y_t \sim P$$

$D \longrightarrow D$

KL divergence

KL divergence rate

$$\lim_{t\to\infty} \frac{1}{t} D\left( Y_1, \ldots, Y_t \sim P_\theta \middle\| Y'_1, \ldots, Y'_t \sim P_{\theta'} \right) = \mathbb{E}_{(Y,Y')\sim Q_\theta} \log \frac{P_\theta(Y,Y')}{P_{\theta'}(Y,Y')}$$

$$\triangleq D\left( P_\theta \middle\| P_{\theta'} \right)$$

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Distributions — $\mathcal{P}(\mathcal{Y})$

Markov chains — $\mathcal{W}(\mathcal{Y}, \mathcal{E})$

$$\nabla^{(e)}, \nabla^{(m)} \xrightarrow{\text{limit } \infty} \nabla^{(e)}, \nabla^{(m)}$$

$$\mathfrak{g} \longrightarrow \mathfrak{g}$$

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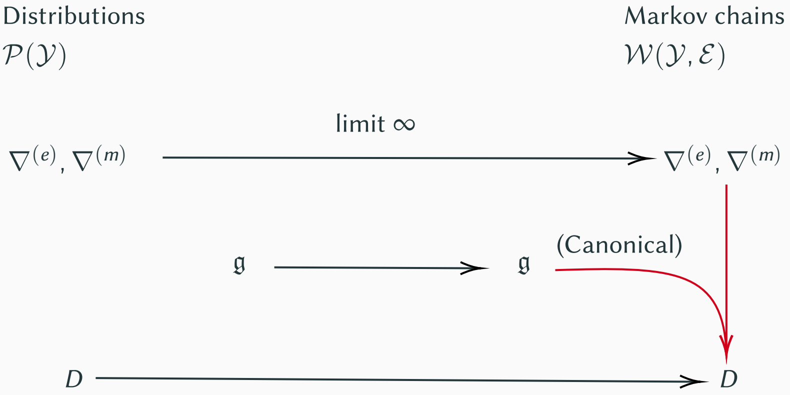

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View $\mathcal{W}(\mathcal{Y}, \mathcal{E})$ as a smooth manifold.

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**Fisher information metric $\mathfrak{g}$**

$$\mathfrak{g}_{ij}(P_\theta) \triangleq \sum_{(y,y') \in \mathcal{E}} Q_\theta(y, y') \partial_i \log P_\theta(y, y') \partial_j \log P_\theta(y, y').$$

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# Exponential families of transition kernels (Nagaoka, 2005)

Let $\Theta \subseteq \mathbb{R}^d$, open connected parameter space.

$$\mathcal{V}_e = \left\{ P_\theta \colon \theta = (\theta^1, \ldots, \theta^d) \in \Theta \right\} \subset \mathcal{W}$$

is an e-family with natural parameter $\theta$, whenever there exist functions $K, R_\theta, \psi_\theta, g_1, \ldots, g_d$ such that

$$\log P_\theta(y, y') = K(y, y') + \sum_{i=1}^{d} \theta^i g_i(y, y') + \underbrace{R_\theta(y') - R_\theta(y) - \psi_\theta}_{\text{rescaling terms}}.$$

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**Example 1 (Nagaoka, 2005)**

$\mathcal{W}(\mathcal{Y}, \mathcal{E})$ forms an e-family of dimension $|\mathcal{E}| - |\mathcal{Y}|$ . For $\mathcal{E} = \mathcal{Y}^2$,

$$\log P(y, y') = \sum_{i=1}^{|\mathcal{Y}|} \sum_{\substack{j=1 \\ j \neq y_\star}}^{|\mathcal{Y}|} \overbrace{\log \frac{P(i, j) P(j, y_\star)}{P(i, y_\star) P(y_\star, y_\star)}}^{\theta^{ij}} \overbrace{\delta_i(y) \delta_j(y')}^{g_{ij}(y, y')}$$

$$+ \log P(y, y_\star) - \log P(y', y_\star) + \log P(y_\star, y_\star).$$

# Mixture families (Nagaoka, 2005)

We say that $\mathcal{V}_m$ is a mixture family when there exists $C, F_1, \dots, F_d \in \mathcal{F}$, such that $C, C + F_1, \dots, C + F_d$ are affinely independent,

$$\sum_{y,y' \in \mathcal{Y}} C(y, y') = 1, \qquad \sum_{y,y' \in \mathcal{Y}} F_i(y, y') = 0, \forall i \in [d],$$

and

$$\mathcal{V}_m = \left\{ P_\xi \in \mathcal{W} \colon Q_\xi = C + \sum_{i=1}^{d} \xi^i F_i, \xi \in \Xi \right\}$$

where $\Xi = \left\{ \xi \in \mathbb{R}^d \colon Q_\xi(y, y') > 0, \forall (y, y') \in \mathcal{Y}^2 \right\}$, and $Q_\xi$ is the edge measure that pertains to $P_\xi$.

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**Example 2**

$\mathcal{W}_{\mathrm{rev}}$ (reversible) forms both an m-family and an e-family

(Wolfer and Watanabe, 2021).

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4. Chernoff and Hoeffding bounds with improved pre-factor
   Moulos and Anantharam (2019).

# Data-processing & Lumping

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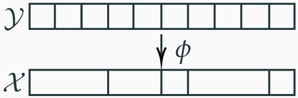
**Lumping**

When $\phi$ is deterministic and $|\mathcal{X}| \leq |\mathcal{Y}|$.



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When $(\kappa(Y_t))_{t \in \mathbb{N}}$ is Markovian, we say $P$ is lumpable: $P \in \mathcal{W}_\kappa(\mathcal{Y}, \mathcal{E})$.

# Lumpability characterization & example

**Characterization (Kemeny and Snell, 1983, Theorem 6.3.2)**

$P \in \mathcal{W}_\kappa(\mathcal{Y}, \mathcal{E})$ iff for all $x, x' \in \mathcal{X}$, and for all $y_1, y_2 \in \mathcal{S}_x$,

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$\mathcal{Y} = \{0, 1, 2\}, \mathcal{X} = \{0, 1\}$

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# Markov Embeddings

# Markov morphisms – Distribution setting

**Definition 4 (Čencov (1978); Campbell (1986))**

Let the partition $\biguplus_{x \in \mathcal{X}} \mathcal{S}_x = \mathcal{Y}$.

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# Markov morphisms – Distribution setting

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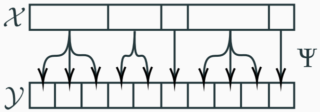

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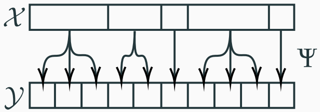


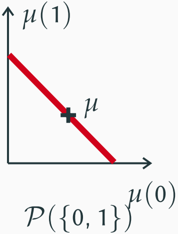


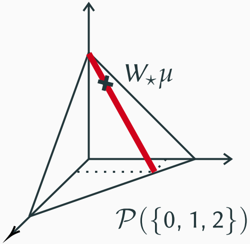

# Congruent linear embeddings

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For a statistic $\kappa\colon \mathcal{Y} \to \mathcal{X}$, a $\kappa$-congruent embedding $K_\star$ is a map $K_\star\colon \mathbb{R}^{\mathcal{X}} \to \mathbb{R}^{\mathcal{Y}}$ that verifies:

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**Property (Ay et al., 2017, Example 5.2)**

Congruent embedding iff (congruent) Markov morphism.

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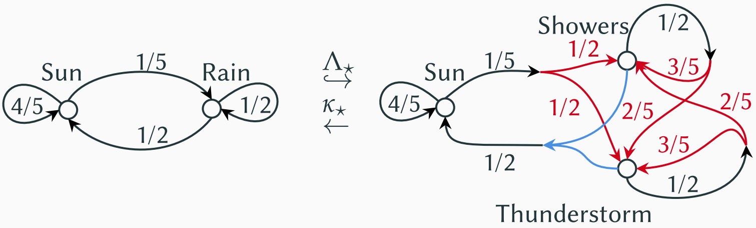
$$\Lambda = \begin{pmatrix} W_{1,1} & W_{1,2} & \cdots & & W_{1,n} \\ \vdots & & & & \vdots \\ W_{x,1} & \cdots & W_{x,x'} & \cdots & W_{x,n} \\ \vdots & & & & \vdots \\ W_{n,1} & & \cdots & & W_{n,n} \end{pmatrix}.$$

# Example: weather model




$$P = \begin{pmatrix} 4/5 & 1/5 \\ 1/2 & 1/2 \end{pmatrix}, \Lambda = \left(\begin{array}{c:cc} 1 & 1/2 & 1/2 \\ \hdashline 1 & 3/5 & 2/5 \\ 1 & 2/5 & 3/5 \end{array}\right),$$

$$\Lambda_\star P = \left(\begin{array}{c:cc} 4/5 & 1/10 & 1/10 \\ \hdashline 1/2 & 3/10 & 1/5 \\ 1/2 & 1/5 & 3/10 \end{array}\right).$$

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**Important observation**
Unlike distribution setting, Markov embeddings are not m-geodesic affine.

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# Congruent Embeddings are Compatible Markov Embeddings

**Theorem 10**
*Let* $(\mathcal{X}, \mathcal{D})$, $(\mathcal{Y}, \mathcal{E})$ *be strongly connected digraphs, and* $\kappa \colon \mathcal{Y} \to \mathcal{X}$ *a lumping function, such that* $\kappa_2(\mathcal{E}) = \mathcal{D}$, *and* $\mathcal{W}_\kappa(\mathcal{Y}, \mathcal{E}) \neq \emptyset$.

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*$K_\star \colon \mathcal{W}(\mathcal{X}, \mathcal{D}) \to \mathcal{W}_\kappa(\mathcal{Y}, \mathcal{E})$ is a $\kappa$-congruent embedding,*

*if and only if*

*$K_\star$ is a $\kappa$-compatible Markov embedding.*

# Example: Hudson expansions

# Hudson expansions

Natural expansions, inverse of lumping, considered in Kemeny and Snell (1983, Section 6.5,p.140).

$$H_\star \colon \mathcal{W}(\mathcal{X}, \mathcal{D}) \to \mathcal{W}_h(\mathcal{D}, H_{\mathcal{D}}).$$

When

$$X_1, X_2, \dots, X_t, \dots \sim \bar{P} \in \mathcal{W}(\mathcal{X}, \mathcal{D}),$$

then sliding window observations

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(iv) *Theory extends to higher-order.*

# Information Geometry of Lumpable Kernels

# Linear family of kernels that lump into prescribed $\bar{P}_0$

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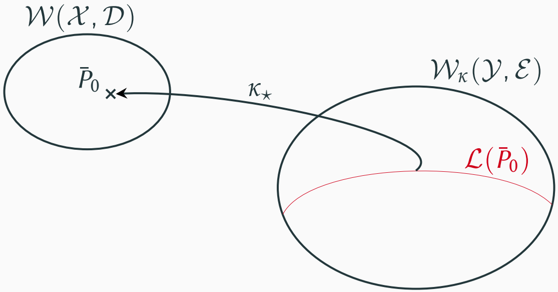

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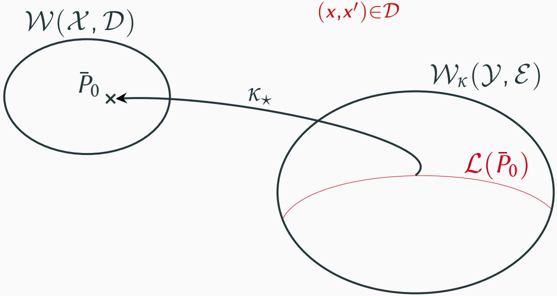
**Lemma 12**
*$\mathcal{L}(\bar{P}_0)$ forms an m-family in $\mathcal{W}(\mathcal{Y}, \mathcal{E})$, with*

$$\dim \mathcal{L}(\bar{P}_0) = |\mathcal{E}| - \sum_{(x,x') \in \mathcal{D}} |\mathcal{S}_x|.$$



# Application: Maximum Entropy Principle

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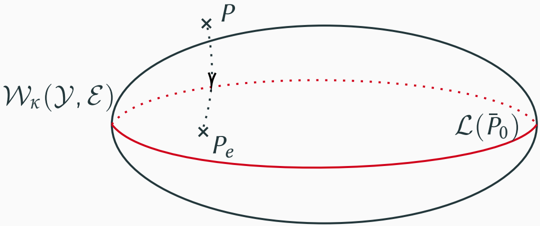

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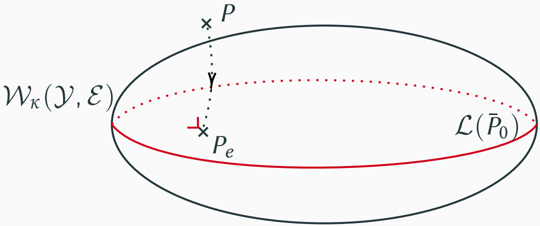

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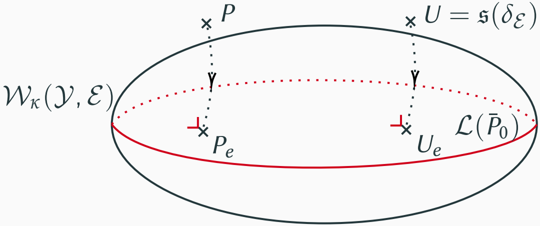

# Application: Maximum Entropy Principle

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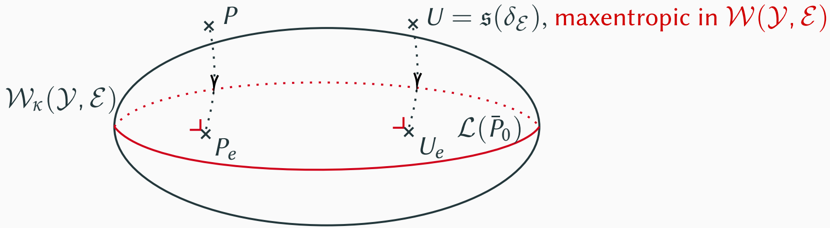
e-geodesic going through $P$ and $P_e$ is $\perp$ to $\mathcal{L}(\bar{P}_0)$.



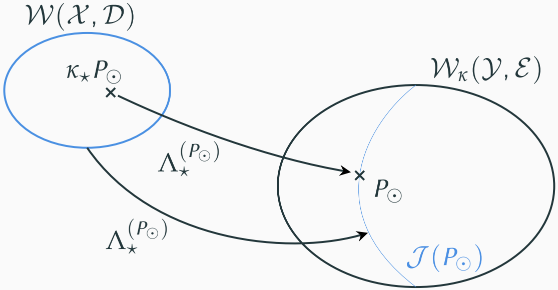

Entropy rate: $H(P) \triangleq \lim_{k \to \infty} \frac{1}{k} H(Y_1, Y_2, \dots, Y_k)$

and rewrite

$$U_e = \arg\min_{P' \in \mathcal{L}(\bar{P}_0)} \left\{ -H(P') - \overbrace{\mathbb{E}_{(Y,Y') \sim Q'} \left[ \log U(Y, Y') \right]}^{-\log \rho(\mathfrak{s}(\delta_{\mathcal{E}}))} \right\} = \arg\max_{P' \in \mathcal{L}(\bar{P}_0)} H(P').$$

# e-family of embedded kernels at some prescribed origin $P_\odot$

$P_\odot \in \mathcal{W}_\kappa(\mathcal{Y}, \mathcal{E})$, and $\Lambda_\star^{(P_\odot)}$ canonical emb. (e.g. $\Lambda_\star^{(P_\odot)} \kappa_\star P_\odot = P_\odot$).



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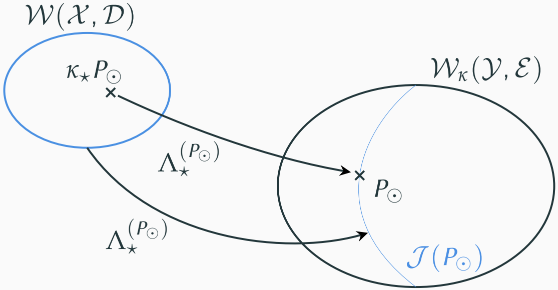
$$\mathcal{J}(P_\odot) \triangleq \left\{ \Lambda_\star^{(P_\odot)} \bar{P} \colon \bar{P} \in \mathcal{W}(\mathcal{X}, \mathcal{D}) \right\} \subset \mathcal{W}_\kappa(\mathcal{Y}, \mathcal{E}).$$



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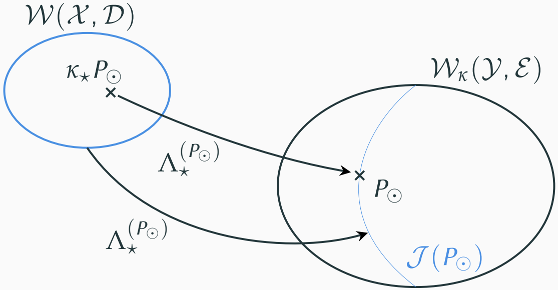
**Lemma 13**
*$\mathcal{J}(P_\odot)$ forms an e-family in $\mathcal{W}(\mathcal{Y}, \mathcal{E})$, with*

$$\dim \mathcal{J}(P_\odot) = |\mathcal{D}| - |\mathcal{X}|.$$



# Foliated manifold of lumpable kernels

**Theorem 14**

*For any fixed* $\bar{P}_0 \in \mathcal{W}(\mathcal{X}, \mathcal{D})$,

$$\mathcal{W}_\kappa(\mathcal{Y}, \mathcal{E}) = \biguplus_{P_\odot \in \mathcal{L}(\bar{P}_0)} \mathcal{J}(P_\odot).$$

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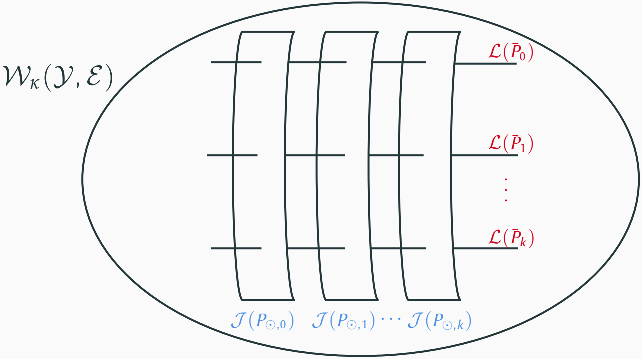


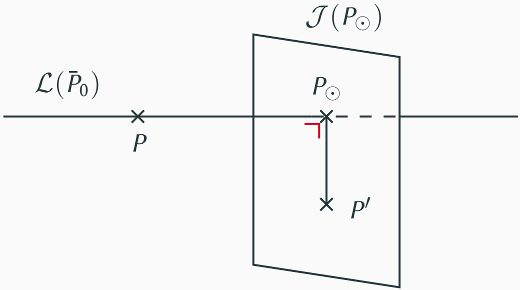
# Application: Leaf projection

**Theorem 15 (Pythagorean identity)**
*Fix* $\bar{P}_0 \in \mathcal{W}(\mathcal{X}, \mathcal{D})$. *Let* $P_\odot, P \in \mathcal{L}(\bar{P}_0)$ *and* $P' \in \mathcal{J}(P_\odot)$.

$$D\left(P \| P'\right) = D\left(P \| P_\odot\right) + D\left(P_\odot \| P'\right),$$

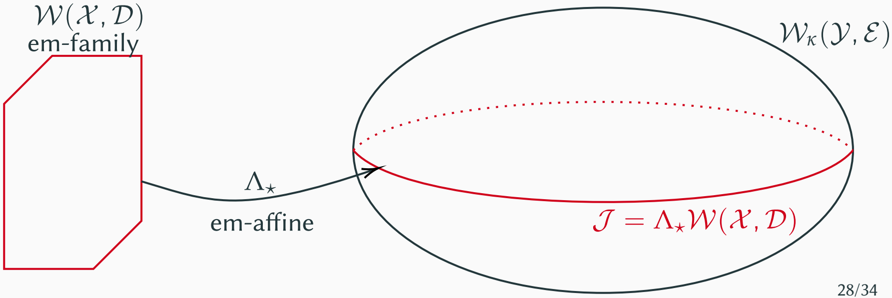
*and* $P_\odot$ *verifies*

$$P_\odot = \underset{P'' \in \mathcal{L}(\bar{P}_0)}{\arg\min}\; D\left(P'' \| P'\right) = \underset{P'' \in \mathcal{J}(P_\odot)}{\arg\min}\; D\left(P \| P''\right).$$



$\mathcal{W}(\mathcal{X}, \mathcal{D})$ em-family, and $\Lambda_\star \colon \mathcal{W}(\mathcal{X}, \mathcal{D}) \to \mathcal{W}_\kappa(\mathcal{Y}, \mathcal{E})$, em-affine.

Then $\mathcal{J} \triangleq \{\Lambda_\star \bar{P} \colon \bar{P} \in \mathcal{W}(\mathcal{X}, \mathcal{D})\}$, is em-family,



# Data-processing inequality & m-contraction

$\mathcal{W}(\mathcal{X}, \mathcal{D})$ em-family, and $\Lambda_\star \colon \mathcal{W}(\mathcal{X}, \mathcal{D}) \to \mathcal{W}_\kappa(\mathcal{Y}, \mathcal{E})$, em-affine.

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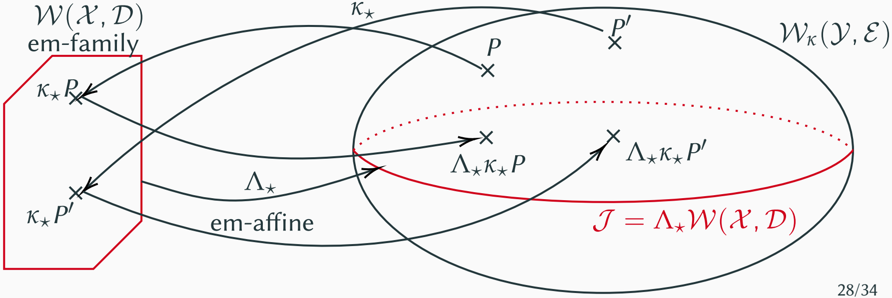

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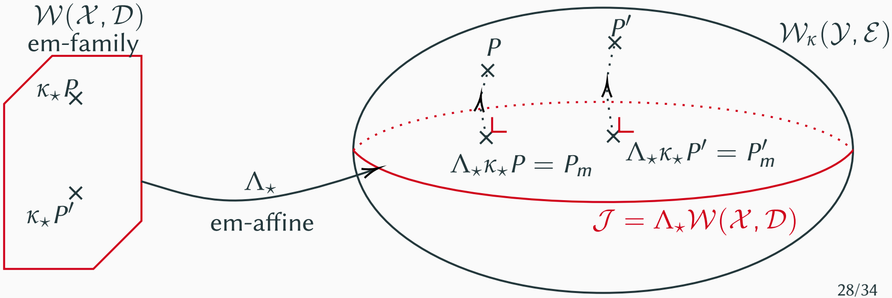
For any $P \in \mathcal{W}_\kappa(\mathcal{Y}, \mathcal{E})$, $P_m \triangleq \arg\min_{\tilde{P} \in \mathcal{J}} D\left(P \| \tilde{P}\right) \overset{\text{Lemma}}{=} \Lambda_\star \kappa_\star P$.



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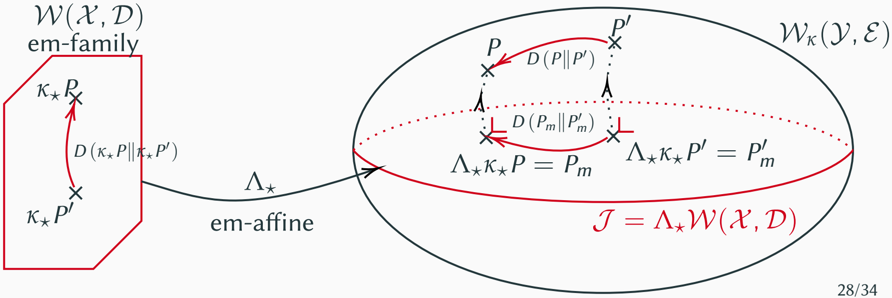
$$D\left(P \| P'\right) \geq D\left(\Lambda_\star \kappa_\star P \| \Lambda_\star \kappa_\star P'\right) = D\left(\kappa_\star P \| \kappa_\star P'\right).$$



**Thank you for listening!**

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