Test-measured Rényi divergences

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MOMENTUM OF INNOVATION

Lendület program

Measured Rényi divergences

Classical Rényi divergences: ϱ, σ probability distributions on a finite set Ω

$$D_\alpha^{\mathrm{cl}}(\varrho\|\sigma) := \begin{cases} \frac{1}{\alpha-1}\log\sum_{\omega\in\Omega}\varrho(\omega)^\alpha\sigma(\omega)^{1-\alpha}, & \alpha\in(0,1) \text{ or } \operatorname{supp}\varrho\subseteq\operatorname{supp}\sigma, \\ +\infty, & \alpha>1, \quad \operatorname{supp}\varrho\not\subseteq\operatorname{supp}\sigma \end{cases}$$

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Measured Rényi divergences: ϱ, σ finite-dimensional density operators

$$D_\alpha^{\mathrm{meas}}(\varrho\|\sigma) := \sup\{D_\alpha^{\mathrm{cl}}(\mathcal{M}(\varrho)\|\mathcal{M}(\sigma)) : M\in\mathrm{POVM}(\mathcal{H},[m]),\, m\in\mathbb{N}\}.$$

$$\mathcal{M}(\cdot) := (\operatorname{Tr} M_i(\cdot))_{i\in[m]}$$

- Smallest quantum Rényi divergences that are monotone under CPTP maps.
- Not additive on tensor products.

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- Post-measurement Rényi divergences in a binary state discrimination problem.

Regularized measured Rényi divergences

Regularized measured Rényi α-divergence

$$\overline{D}_{\alpha}^{\mathrm{meas}}(\varrho\|\sigma) := \sup_{n\in\mathbb{N}} \frac{1}{n} D_{\alpha}^{\mathrm{meas}}(\varrho^{\otimes n}\|\sigma^{\otimes n}) = \lim_{n\to+\infty} \frac{1}{n} D_{\alpha}^{\mathrm{meas}}(\varrho^{\otimes n}\|\sigma^{\otimes n}),$$

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  $\Longrightarrow M_{(i,j)} := M_i^{(1)} \otimes M_j^{(2)}$ POVM on $\mathcal{H}^{\otimes (n_1+n_2)}$ with $m_1 \cdot m_2$ outcomes

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super-additivity of $n \mapsto D_\alpha^{\text{test}}(\varrho^{\otimes n}\|\sigma^{\otimes n})$ is not clear; in fact, it is not true for $\alpha < 1$.

Some relations

$D_{\alpha,z}(\varrho\|\sigma) := \frac{\log \operatorname{Tr}\left(\varrho^{\frac{\alpha}{2z}}\sigma^{\frac{1-\alpha}{z}}\varrho^{\frac{\alpha}{2z}}\right)^z}{\alpha-1}$ Rényi (α, z)-divergences[1]

$D_\alpha = D_{\alpha,1}$: Petz-type Rényi divergences[2]

$D_{\alpha,\alpha}$: sandwiched Rényi divergences[3]

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\begin{array}{ccccc}
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 & & & \le^{\mathbf{3}} & D_\alpha
\end{array}
$$

[1] Audenaert,Datta 2013; Jaksic et al. 2010; [2] Petz 1985; [3] Müller-Lennert et al. & Wilde et al. 2013
[4] M., Ogawa 2015, [5] Hayashi, Tomamichel 2016, [6] Vrana 2022

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All the red quantities are equal for commuting states.

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All the green quantities are equal for $\alpha > 1$.

Optimal test sequence for the strong converse exponent
of asymptotic i.i.d. binary state discrimination.[4]

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All the green quantities are equal for $\alpha > 1$.

Note: Attaining $D_\alpha^{\mathrm{meas}}(\varrho^{\otimes n}\|\sigma^{\otimes n})$ generally requires
a number of measurement outcomes that diverges in n.

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All the green quantities are equal for $\alpha > 1$.

But not for $\alpha < 1$.[7]

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Operational interpretations

ϱ, σ density operators

$$\beta_{e^{-nr}}(\varrho^{\otimes n}\|\sigma^{\otimes n}) := \min\{\operatorname{Tr}\varrho^{\otimes n}(I-T_n) : T \in \mathbb{T}(\mathcal{H}^{\otimes n}),\ \operatorname{Tr}\sigma^{\otimes n}T_n \le e^{-nr}\}$$

$\alpha > 1$:

$$\sup_{\alpha>1} \frac{\alpha-1}{\alpha}[r - \overline{D}_\alpha^{\text{test}}(\varrho\|\sigma)] =^{\mathbf{1}} \lim_{n\to+\infty} -\frac{1}{n}\log(1-\beta_{e^{-nr}}(\varrho^{\otimes n}\|\sigma^{\otimes n}))$$

$$\overline{D}_\alpha^{\text{test}}(\varrho\|\sigma) =^{\mathbf{1}} \min\left\{r_0 : \lim_{n\to+\infty} -\frac{1}{n}\log(1-\beta_{e^{-nr}}(\varrho^{\otimes n}\|\sigma^{\otimes n})) \ge \frac{\alpha-1}{\alpha}(r-r_0)\right\}$$

$\alpha < 1$:

$$H_r(\varrho\|\sigma) := \sup_{\alpha\in(0,1)} \frac{\alpha-1}{\alpha}[r - D_\alpha(\varrho\|\sigma)] =^{\mathbf{2}} \lim_{n\to+\infty} -\frac{1}{n}\log\beta_{e^{-nr}}(\varrho^{\otimes n}\|\sigma^{\otimes n})$$

$$\sup_{\alpha\in(0,1)}\{c\alpha - (\alpha-1)D_\alpha(\varrho\|\sigma)\} =^{\mathbf{2}} \lim_{n\to+\infty} -\frac{1}{n}\log \min_{T\in\mathbb{T}(\mathcal{H}^{\otimes n})}\{e^{-nc}\operatorname{Tr}\varrho^{\otimes n}(I-T) + \operatorname{Tr}\sigma^{\otimes n}T\}$$

$$\overline{D}_\alpha^{\text{test}}(\varrho\|\sigma) =^{\mathbf{3}} \lim_{n\to+\infty} -\frac{1}{n}\log \min_{T\in\mathbb{T}(\mathcal{H}^{\otimes n})}\left\{\left(\operatorname{Tr}\varrho^{\otimes n}(I-T)\right)^{\frac{\alpha}{1-\alpha}} + \operatorname{Tr}\sigma^{\otimes n}T\right\}$$

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[1] M., Ogawa 2013; [2] Hayashi 2006 & Nagaoka 2006; [3] M., Hiai 2022

Single-copy expression

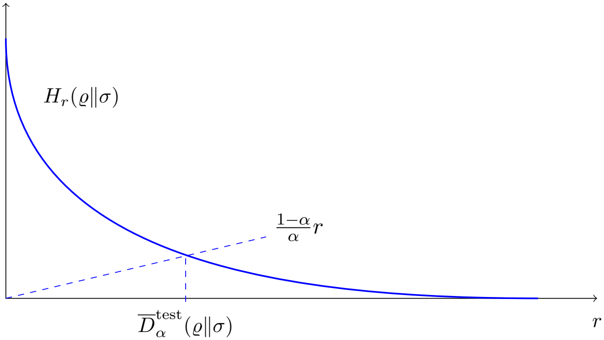
$$\overline{D}_\alpha^{\text{test}}(\varrho\|\sigma) =^1 \lim_{n\to+\infty} -\frac{1}{n}\log \min_{T\in\mathbb{T}(\mathcal{H}^{\otimes n})}\left\{\left(\operatorname{Tr}\varrho^{\otimes n}(I-T)\right)^{\frac{\alpha}{1-\alpha}} + \operatorname{Tr}\sigma^{\otimes n}T\right\}$$

[1] M., Hiai 2022; [2] Salzmann, Datta, 2021

Single-copy expression

$$\overline{D}_\alpha^{\text{test}}(\varrho\|\sigma) \overset{1}{=} \lim_{n\to+\infty} -\frac{1}{n}\log \min_{T\in\mathbb{T}(\mathcal{H}^{\otimes n})}\left\{\left(\operatorname{Tr}\varrho^{\otimes n}(I-T)\right)^{\frac{\alpha}{1-\alpha}} + \operatorname{Tr}\sigma^{\otimes n}T\right\}$$

$$\overset{1,2}{=} \sup\left\{r\geq 0:\ H_r(\varrho\|\sigma)\geq \frac{1-\alpha}{\alpha}r\right\}$$



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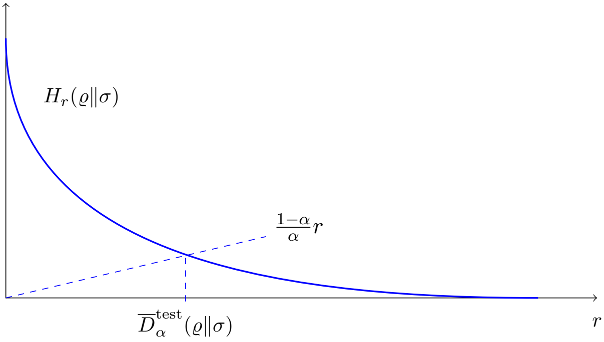
$$=^{1,2} \sup\left\{r\geq 0:\ H_r(\varrho\|\sigma)\geq \frac{1-\alpha}{\alpha}r\right\}$$

$$=^{2} \alpha \sup_{0<t<1}\frac{(t-1)D_t(\varrho\|\sigma)}{t(2\alpha-1)-\alpha}$$



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\begin{aligned}
\overline{D}_\alpha^{\text{test}}(\varrho\|\sigma) &=^{1} \lim_{n\to+\infty} -\frac{1}{n}\log \min_{T\in\mathbb{T}(\mathcal{H}^{\otimes n})} \left\{ \left(\operatorname{Tr}\varrho^{\otimes n}(I-T)\right)^{\frac{\alpha}{1-\alpha}} + \operatorname{Tr}\sigma^{\otimes n}T \right\} \\
&=^{1,2} \sup\left\{ r\ge 0:\ H_r(\varrho\|\sigma) \ge \frac{1-\alpha}{\alpha}r \right\} \\
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\end{aligned}
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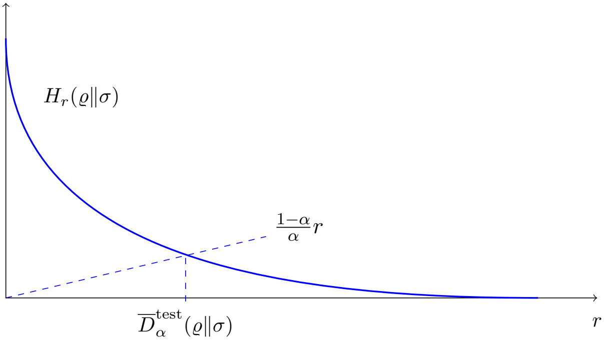
Not explicit, but single-copy.



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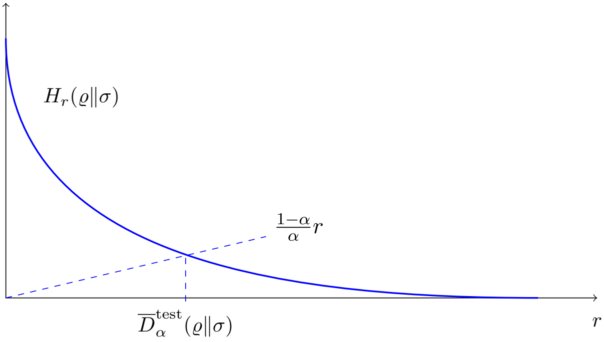
Not explicit, but single-copy. Works also in general von Neumann algebras.[1]



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Regularized test-measured and Petz-type comparison

$$\overline{D}_{\alpha}^{\text{test}}(\varrho\|\sigma) \quad = \quad \sup\left\{r \geq 0 : \; H_r(\varrho\|\sigma) \geq \frac{1-\alpha}{\alpha} r\right\}$$

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Theorem:[1] $\varrho = \sum_{i=1}^k r_i P_i, \qquad \sigma = \sum_{i=1}^m s_i Q_i$

$$\frac{1}{2} D_\alpha(\varrho\|\sigma) \leq \overline{D}_\alpha^{\text{test}}(\varrho\|\sigma) \leq D_\alpha(\varrho\|\sigma), \qquad \alpha \in (0,1).$$

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$$\frac{1}{2} D_\alpha(\varrho\|\sigma) \leq \overline{D}_\alpha^{\text{test}}(\varrho\|\sigma) \leq D_\alpha(\varrho\|\sigma), \qquad \alpha \in (0,1).$$

The following are equivalent:

1. $\overline{D}_\alpha^{\text{test}}(\varrho\|\sigma) = D_\alpha(\varrho\|\sigma)$ for some $\alpha \in (0,1)$.
2. $\overline{D}_\alpha^{\text{test}}(\varrho\|\sigma) = D_\alpha(\varrho\|\sigma)$ for every $\alpha \in (0,1)$.
3. $\frac{1}{n} D_\alpha^{\text{test}}(\varrho^{\otimes n}\|\sigma^{\otimes n}) = D_\alpha(\varrho\|\sigma)$ for every $\alpha \in (0,1)$ and every $n \in \mathbb{N}$.
4. a) $\exists\, \kappa > 0$ and projections $P_i' \leq Q_i$: $\varrho = \sum_{i=1}^m \kappa s_i P_i'$, or
   b) $\exists\, \eta > 0$ and projections $Q_i' \leq P_i$: $\sigma = \sum_{i=1}^k \eta r_i Q_i'$.

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Corollary: $\overline{D}_\alpha^{\text{test}}(\varrho\|\sigma) < D_\alpha(\varrho\|\sigma)$ in general

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Corollary: $\overline{D}_\alpha^{\text{test}}(\varrho\|\sigma) < D_\alpha(\varrho\|\sigma)$ in general, even for commuting states.

[1] M., Hiai 2022

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1. $\overline{D}_\alpha^{\text{test}}(\varrho\|\sigma)=D_\alpha(\varrho\|\sigma)$ for some $\alpha\in(0,1)$.
2. $\overline{D}_\alpha^{\text{test}}(\varrho\|\sigma)=D_\alpha(\varrho\|\sigma)$ for every $\alpha\in(0,1)$.
3. $\frac{1}{n}D_\alpha^{\text{test}}(\varrho^{\otimes n}\|\sigma^{\otimes n})=D_\alpha(\varrho\|\sigma)$ for every $\alpha\in(0,1)$ and every $n\in\mathbb{N}$.
4. a) $\exists\,\kappa>0$ and projections $P_i'\le Q_i$: $\varrho=\sum_{i=1}^m \kappa s_iP_i'$, or
   b) $\exists\,\eta>0$ and projections $Q_i'\le P_i$: $\sigma=\sum_{i=1}^k \eta r_iQ_i'$.

Corollary: $\overline{D}_\alpha^{\text{test}}(\varrho\|\sigma)<D_\alpha(\varrho\|\sigma)$ in general, even for commuting states.

Corollary: $\overline{D}_\alpha^{\text{test}}$ is not a quantum extension of the classical Rényi α-divergence for any $\alpha\in(0,1)$.

[1] M., Hiai 2022

Regularized test-measured and Petz-type comparison

$$\overline{D}_\alpha^{\text{test}}(\varrho\|\sigma) \quad = \quad \sup\left\{r \geq 0 : \; H_r(\varrho\|\sigma) \geq \frac{1-\alpha}{\alpha} r\right\}$$

Theorem:[1] $\varrho = \sum_{i=1}^k r_i P_i, \qquad \sigma = \sum_{i=1}^m s_i Q_i$

$$\frac{1}{2} D_\alpha(\varrho\|\sigma) \leq \overline{D}_\alpha^{\text{test}}(\varrho\|\sigma) \leq D_\alpha(\varrho\|\sigma), \qquad \alpha \in (0,1).$$

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Corollary: $\forall \alpha, \beta \in (0,1)$

$$D_\alpha(\varrho\|\sigma) \geq \frac{\alpha(1-\beta)}{\alpha - 2\alpha\beta + \beta} D_\beta(\varrho\|\sigma)$$

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$$D_\alpha(\varrho\|\sigma)\ge \frac{\alpha(1-\beta)}{\alpha-2\alpha\beta+\beta}D_\beta(\varrho\|\sigma)$$

Corollary:

$$\lim_{\alpha\searrow 0}\overline{D}_\alpha^{\rm test}(\varrho\|\sigma)=D_0(\varrho\|\sigma),$$

$$\overline{D}_{1/2}^{\rm test}(\varrho\|\sigma)=C(\varrho\|\sigma):=-\min_{\alpha\in[0,1]}\log\operatorname{Tr}\varrho^\alpha\sigma^{1-\alpha},$$ Chernoff divergence

$$\lim_{\alpha\to 1}\overline{D}_\alpha^{\rm test}(\varrho\|\sigma)=D(\varrho\|\sigma).$$ Umegaki relative entropy

[1] M., Hiai 2022

Regularized test-measured and Petz-type comparison

$$\overline{D}_\alpha^{\text{test}}(\varrho\|\sigma) \quad := \quad \limsup_{n\to+\infty} \frac{1}{n} D_\alpha^{\text{test}}\left(\varrho^{\otimes n}\|\sigma^{\otimes n}\right) < D_\alpha(\varrho\|\sigma),$$

Regularized test-measured and Petz-type comparison

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$$\wedge| \qquad \| \ ?$$

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Lemma

$$\left.\begin{array}{r} a_n < a, \quad n\in\mathbb{N} \\ \limsup_{n\to+\infty} a_n < a \end{array}\right\} \iff \sup_{n\in\mathbb{N}} a_n < a$$

Regularized test-measured and Petz-type comparison

Proposition:[1]

$$\frac{1}{n}D_\alpha^{\mathrm{test}}(\varrho^{\otimes n}\|\sigma^{\otimes n}) \le \frac{1}{n}D_\alpha^{\mathrm{meas}}(\varrho^{\otimes n}\|\sigma^{\otimes n}) \le \left\{ \begin{array}{ll} \overline{D}_\alpha^{\mathrm{meas}}(\varrho\|\sigma), & \alpha\in(0,1)\setminus\{1/2\} \\ D_\alpha(\varrho\|\sigma), & \alpha=1/2 \end{array} \right\}$$
$$\le D_\alpha(\varrho\|\sigma).$$

Moreover, if one of the following holds, then ϱ and σ commute:

1. $\varrho^0 \le \sigma^0$, $\frac{1}{n}D_\alpha^{\mathrm{meas}}(\varrho^{\otimes n}\|\sigma^{\otimes n}) = \overline{D}_\alpha^{\mathrm{meas}}(\varrho\|\sigma)$ for some $\alpha\in(\frac{1}{2},1)$ and $n\in\mathbb{N}$;
2. $\varrho^0 \ge \sigma^0$, $\frac{1}{n}D_\alpha^{\mathrm{meas}}(\varrho^{\otimes n}\|\sigma^{\otimes n}) = \overline{D}_\alpha^{\mathrm{meas}}(\varrho\|\sigma)$ for some $\alpha\in(0,\frac{1}{2})$ and $n\in\mathbb{N}$;
3. $\varrho^0 \le \sigma^0$ or $\varrho^0 \ge \sigma^0$, $\frac{1}{n}D_{1/2}^{\mathrm{meas}}(\varrho^{\otimes n}\|\sigma^{\otimes n}) = D_{1/2}(\varrho\|\sigma)$ for some $n\in\mathbb{N}$.

[1] M., Hiai 2022; Berta, Fawzi, Tomamichel 2015

Regularized test-measured and Petz-type comparison

Theorem: $\varrho = \sum_{\omega\in\Omega} \varrho(\omega)\,|\omega\rangle\langle\omega| \;\neq\; \sigma = \sum_{\omega\in\Omega} \sigma(\omega)\,|\omega\rangle\langle\omega|, \quad \varrho^0 = \sigma^0.$

T.f.a.e.:

1. $\hat{D}_\alpha^{\mathrm{test}}(\varrho\|\sigma) = D_\alpha(\varrho\|\sigma)$ for every/some $\alpha \in (0,1)$;
2. $D_\alpha^{\mathrm{test}}(\varrho\|\sigma) = D_\alpha(\varrho\|\sigma)$ for every/some $\alpha \in (0,1)$;
3. there exist some $\Omega_0 \subseteq \Omega$ and strictly positive constants c_0, c_1 such that

$$\varrho(\omega) = c_0\sigma(\omega), \quad \omega \in \Omega_0, \qquad \varrho(\omega) = c_1\sigma(\omega), \quad \omega \in \Omega \setminus \Omega_0.$$

In particular, if 3. is not satisfied then $\hat{D}_\alpha^{\mathrm{test}}(\varrho\|\sigma) < D_\alpha(\varrho\|\sigma)$ for every $\alpha \in (0,1)$.

Corollary: $\hat{D}_\alpha$ is not a quantum Rényi α-divergence for any $\alpha \in (0,1)$.

Corollary: If 3. above holds with $c_0 \neq c_1$ then

$$\overline{D}_\alpha^{\mathrm{test}}(\varrho\|\sigma) < \hat{D}_\alpha^{\mathrm{test}}(\varrho\|\sigma) = D_\alpha(\varrho\|\sigma).$$