Continuity of quantum entropic quantities via almost convexity

BEYONDIID10

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Outline

From almost convexity to continuity bounds

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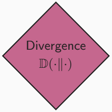

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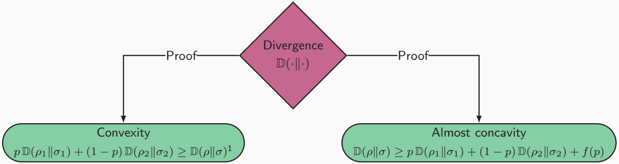

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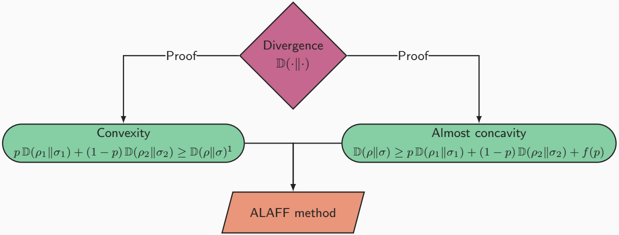

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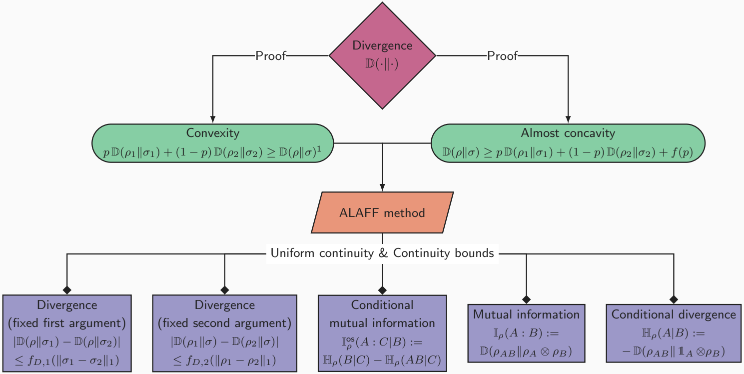

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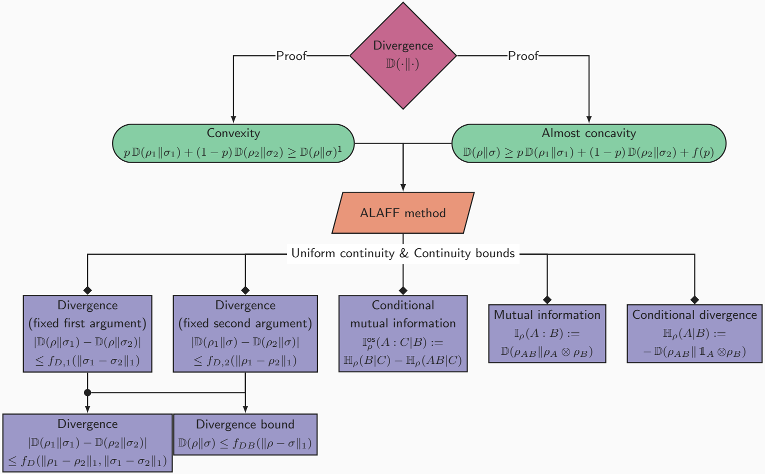

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The ALAFF method

Definition (Almost locally affine function)

f real-valued function on the convex set $\mathcal{S}_0 \subseteq \mathcal{S}(\mathcal{H})$. It is called *almost locally affine (ALAFF)*, if

$$-a_f(p) \leq f(p\rho + (1-p)\sigma) - pf(\rho) - (1-p)f(\sigma) \leq b_f(p)$$

for all $p \in [0,1]$ and $\rho, \sigma \in \mathcal{S}_0$, with $a_f : [0,1] \to \mathbb{R}$ and $b_f : [0,1] \to \mathbb{R}$, continuous, non decreasing on $[0,1/2]$ and vanishing for $p \to 0^+$.

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Definition (Perturbed Δ-invariant subset)

Let $s \in [0,1)$. $\mathcal{S}_0 \subseteq \mathcal{S}(\mathcal{H})$ is called *s-perturbed Δ-invariant*, if for $\rho, \sigma \in \mathcal{S}_0$, $\rho \neq \sigma$ there exists $\tau \in \mathcal{S}(\mathcal{H})$ such that

$$\Delta^{\pm}(\rho, \sigma, \tau) = s\tau + (1-s)\varepsilon^{-1}[\rho - \sigma]_{\pm} \in \mathcal{S}_0$$

with $\varepsilon := \frac{1}{2}\|\rho - \sigma\|_1$.

The ALAFF method

Theorem (Almost locally affine (ALAFF) method)

$\mathcal{S}_0 \subseteq \mathcal{S}(\mathcal{H})$ be a s-perturbed Δ-invariant convex subset of $\mathcal{S}(\mathcal{H})$ containing more than one element, f an ALAFF function. Then f is uniformly continuous if

$$C_f^s := \sup_{\substack{\rho,\sigma \in \mathcal{S}_0 \\ \frac{1}{2}\|\rho-\sigma\|_1 = 1-s}} |f(\rho) - f(\sigma)| < +\infty \,.$$

In this case, for $\varepsilon \in (0,1]$

$$\sup_{\substack{\rho,\sigma \in \mathcal{S}_0 \\ \frac{1}{2}\|\rho-\sigma\|_1 \leq \varepsilon}} |f(\rho) - f(\sigma)| \leq C_f^s \frac{\varepsilon}{1-s} + \frac{1-s+\varepsilon}{1-s} E_f^{\max}\left(\frac{\varepsilon}{1-s+\varepsilon}\right),$$

with $E_f := a_f + b_f$ and

$$E_f^{\max} : [0,1) \to \mathbb{R}, \qquad p \mapsto (1-p) \max\left\{ \frac{E_f(t)}{1-t} \ : \ 0 \leq t \leq p \right\}.$$

Umegaki relative entropy

$$D(\rho\|\sigma) := \begin{cases} \operatorname{tr}[\rho \log \rho - \rho \log \sigma] & \text{if } \ker \sigma \subseteq \ker \rho \\ +\infty & \text{else} \end{cases}$$

Almost concavity of the relative entropy

Theorem (Almost concavity of the relative entropy)

Let

$(\rho_1, \sigma_1), (\rho_2, \sigma_2) \in \mathcal{S}_{\ker} := \{(\rho, \sigma) \in \mathcal{S}(\mathcal{H}) \times \mathcal{S}(\mathcal{H}) \ : \ \ker \sigma \subseteq \ker \rho\}$ *and* $p \in [0,1]$. *With* $\rho = p\rho_1 + (1-p)\rho_2$ *and* $\sigma = p\sigma_1 + (1-p)\sigma_2$,

$$D(\rho\|\sigma) \geq pD(\rho_1\|\sigma_1) + (1-p)D(\rho_2\|\sigma_2) - h(p)\frac{1}{2}\|\rho_1 - \rho_2\|_1 - f_{c_1,c_2}(p)\,.$$

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$$h(p) = -p\log(p) - (1-p)\log(1-p)\,,$$

$$f_{c_1,c_2}(p) = p\log(p + (1-p)c_1) + (1-p)\log((1-p) + pc_2)\,.$$

The constants in f_{c_1,c_2} *are non-negative real numbers and are given by*

$$c_j := \int_{-\infty}^{\infty} dt \beta_0(t) \operatorname{tr}\left[\rho_j \sigma_j^{\frac{it-1}{2}} \sigma_k \sigma_j^{\frac{-it-1}{2}}\right] < \infty, \qquad j,k = 1,2, \quad j \neq k,$$

with β_0 *a probability density on* $\mathbb{R}$.

Derived continuity and divergence bounds

Quantity	Bound ($\varepsilon \geq \frac{1}{2}\|\rho - \sigma\|_1$)
Conditional entropy	$\lvert H_\rho(A\|B) - H_\sigma(A\|B)\rvert \leq 2\varepsilon \log d_A + (1+\varepsilon) h\left(\frac{\varepsilon}{1+\varepsilon}\right)$
Mutual information	$\lvert I_\rho(A:B) - I_\sigma(A:B)\rvert \leq 2\varepsilon \log \min\{d_A, d_B\} + 2(1+\varepsilon) h\left(\frac{\varepsilon}{1+\varepsilon}\right)$
Conditional mutual information	$\lvert I_\rho(A:B\|C) - I_\sigma(A:B\|C)\rvert \leq 2\varepsilon \log \min\{d_A, d_B\} + 2(1+\varepsilon) h\left(\frac{\varepsilon}{1+\varepsilon}\right)$
Divergence bound	$D(\rho\|\sigma) \leq \varepsilon \log \widetilde{m}_\sigma^{-1} + (1+\varepsilon) h\left(\frac{\varepsilon}{1+\varepsilon}\right)$

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Applications:

- Condition for a state to be an *approximate quantum Markov chain*.
- Continuity bound for the *relative entropy of entanglement*.
- Continuity bound for the *Rains information*.
- Bound on the distance between BS and relative entropy.

Belavkin-Staszewski Entropy

$$\widehat{D}(\rho\|\sigma) := \begin{cases} \mathrm{tr}\big[\rho \log\big(\rho^{1/2}\sigma^{-1}\rho^{1/2}\big)\big] & \text{if } \ker\sigma \subseteq \ker\rho \\ +\infty & \text{else} \end{cases}$$

Almost concavity of the Belavkin-Staszewski entropy

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Let $(\rho_1, \sigma_1), (\rho_2, \sigma_2) \in \mathcal{S}_{\mathrm{ker},+} = \{(\rho, \sigma) \in \mathcal{S}(\mathcal{H}) \times \mathcal{S}(\mathcal{H}) \, : \, \sigma \in \mathcal{S}_+(\mathcal{H})\}$, $p \in [0,1]$. *With* $\rho = p\rho_1 + (1-p)\rho_2$, $\sigma = p\sigma_1 + (1-p)\sigma_2$,

$$\widehat{D}(\rho\|\sigma) \geq p\widehat{D}(\rho_1\|\sigma_1) + (1-p)\widehat{D}(\rho_2\|\sigma_2) - \hat{c}_0(1-\delta_{\rho_1\rho_2})h(p) - f_{\hat{c}_1,\hat{c}_2}(p) \, .$$

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$$h(p) = -p\log(p) - (1-p)\log(1-p)\,,$$

$$f_{\hat{c}_1,\hat{c}_2}(p) = p\log(p + \hat{c}_1(1-p)) + (1-p)\log((1-p) + \hat{c}_2 p)\,,$$

and the constants

$$\hat{c}_0 := \max\{\|\sigma_1^{-1}\|_\infty, \|\sigma_2^{-1}\|_\infty\}\,,$$

$$\hat{c}_j := \int_{-\infty}^{\infty} dt\beta_0(t)\,\mathrm{tr}\Big[\rho_j(\rho_j^{1/2}\sigma_j^{-1}\rho_j^{1/2})^{\frac{it+1}{2}}\rho_j^{-1/2}\sigma_k\rho_j^{-1/2}(\rho_j^{1/2}\sigma_j^{-1}\rho_j^{1/2})^{\frac{-it+1}{2}}\Big],$$

$j, k = 1, 2,,\ j \neq k$ *and* β_0 *probability density on* $\mathbb{R}$.

Taming the BS-entropy

- Definition of the *BS-conditional entropy*

$$\hat{H}_\rho(A|B) := -\widehat{D}(\rho_{AB} \| \mathbb{1}_A \otimes \rho_B) \qquad \widehat{H}^{\mathsf{var}}_\rho(A|B) := \sup_{\sigma_B \in \mathcal{S}(\mathcal{H}_B)} -\widehat{D}(\rho_{AB} \| \mathbb{1}_A \otimes \sigma_B)$$

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- Definition of *BS-conditional mutual information*

$$\widehat{I}_\rho(A:B|C) = \widehat{H}_\rho(A|C) - \widehat{H}_\rho(A|BC)$$

Derived continuity and divergence bounds

Quantity	Bound ($\rho, \sigma \in \mathcal{S}_{\geq m}(\mathcal{H})$, $\varepsilon \geq \frac{1}{2}\|\rho - \sigma\|_1$, $l_m := 1 - md_{\mathcal{H}}$)	Order
BS-conditional entropy	$\lvert\widehat{H}_\rho(A\|B) - \widehat{H}_\sigma(A\|B)\rvert \leq 2l_m^{-1}\varepsilon \log d_A + \frac{l_m+\varepsilon}{l_m}(f_{m^{-1},m^{-1}} + m^{-1}h)\left(\frac{\varepsilon}{l_m+\varepsilon}\right)$	$\sim m^{-1}\sqrt{\varepsilon}$
BS-mutual information	$\lvert\widehat{I}_\rho(A:B) - \widehat{I}_\sigma(A:B)\rvert \leq 2l_m^{-1}\varepsilon(\log\min\{d_A, d_B\} + \log m^{-1}) + \frac{l_m+\varepsilon}{l_m} z_m\left(\frac{\varepsilon}{l_m+\varepsilon}\right)$	$\sim m^{-1}\sqrt{\varepsilon}$
BS-conditional mutual information	$\lvert\widehat{I}_\rho(A:B\|C) - \widehat{I}_\sigma(A:B\|C)\rvert \leq 2\,\varepsilon\, l_m^{-1} \log\min\{d_A, \sqrt{d_{ABC}}\} + 2g_m(\varepsilon)$	$\sim m^{-1}\sqrt{\varepsilon}$
Divergence bound	$\widehat{D}(\rho\\|\sigma) \leq \varepsilon \log m_\sigma^{-1} + (1+\varepsilon)m_\sigma^{-1}h\left(\frac{\varepsilon}{1+\varepsilon}\right)$	$\sim m^{-1}\sqrt{\varepsilon}$

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Applications:

- Continuity bound for the *variational BS-conditional entropy*.
- Continuity bound for the *BS-entropy of entanglement*.
- Continuity bound for the *BS-Rains information*.

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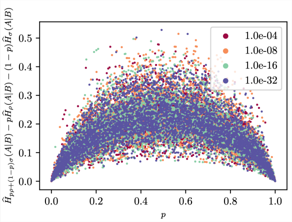
  - → Numerics suggest a bound of the BS-conditional entropy independent of minimal eigenvalues.



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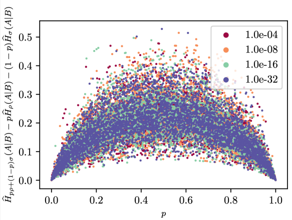
- Understand pathology of BS-entropy.



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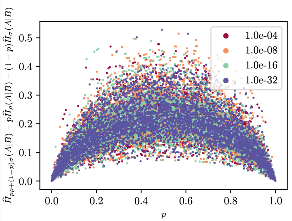

- Understand pathology of BS-entropy.
- Apply method to other divergences.

Summary

- (ALAFF) Method to derive continuity bounds of entropic quantities.
- Recover best-known bounds for the relative entropy in addition to new divergence and continuity bounds.
- Obtain first bounds on BS-conditional entropy, BS-mutual information, BS-conditional mutual information and divergence and continuity bounds for the BS.

Thank you for your attention!