General construction of Unitary t-design from spherical t-design

Masahito Hayashi

Shenzhen Institute for Quantum Science and Engineering,
Southern University of Science and Technology

Graduate School of Mathematics, Nagoya University

Contents

- Spherical t-design, quantum t-design
- Unitary t-design
- Construction of unitary t-design
- 2 Conjectures
- Relation between two conjectures

Spherical t-design, quantum t-design

$S \subset S^d$ is called spherical t-design

when $$\sum_{x \in S} \frac{1}{|S|} f(x) = \int_{S^d} f(x)\mu(dx)$$

for a polynomial f of d+1 variables with degree t.

μ : Haar measure

$S \subset P^d := \left\{ x \in \mathbb{C}^d \,\middle|\, \|x\| = 1 \right\}$ is called

quantum t-design when

$$\sum_{x \in S} \frac{1}{|S|} \left| x^{\otimes t} \right\rangle \left\langle x^{\otimes t} \right| = \int_{P^d} \left| x^{\otimes t} \right\rangle \left\langle x^{\otimes t} \right| \mu(dx)$$

Construction of quantum t-design

Quantum t-design can be constructed from spherical t-design as follows.

$S \subset S^{2d-1}$: spherical 2t-design

$$S' := \left\{ x(v) \right\}_{v \in S} \subset P^d := \left\{ x \in \mathbb{C}^d \,\middle|\, \|1\| = 1 \right\}$$

$$x(v) := \sum_{j=1}^{d} (v_{2j-1} + v_{2j-1} i) e_j$$

S’ is quantum t-design.

Matrix component of $\left| x(v)^{\otimes t} \right\rangle \left\langle x(v)^{\otimes t} \right|$

is a polynomial with degree 2t.

Unitary t-design

$U(d)$: Set of d x d unitary matrices.

$U \subset U(d)$ is called unitary t-design when

$$\sum_{u \in U} \frac{1}{|U|} u^{\otimes t} \otimes \overline{u}^{\otimes t} = \int_{U(d)} u^{\otimes t} \otimes \overline{u}^{\otimes t} \mu(dx)$$

Aim of this talk

We discuss a construction of unitary t-design from spherical 2t-designs as

$$V_{2d-1} \subset S^{2d-1}, V_{2d-3} \subset S^{2d-3}, \ldots, V_3 \subset S^3, V_1 \subset S^1$$

Construction of unitary t-design

Relation between $S^{2d-1}\times S^{2d-3}\times\ldots\times S^{3}\times S^{1}$ and U(d).

Given $v_{2d-1}\in S^{2d-1}, v_{2d-3}\in S^{2d-3},\ldots, v_{3}\in S^{3}, v_{1}\in S^{1}$ we inductively choose an isometry map $f_{v_{2d-1},..,v_{2j+1}}$ from $\mathbb{R}^{2j}$ to subspace of $\mathbb{C}^{d}$ orthogonal to

$$f_{\emptyset}(v_{2d-1}), f_{v_{2d-1}}(v_{2d-3}),.., f_{v_{2d-1},..,v_{2j+3}}(v_{2j+1})$$

We define $u(v_{2d-1},..,v_{1})\in U(d)$ as

$$(f_{\emptyset}(v_{2d-1}), f_{v_{2d-1}}(v_{2d-3}),.., f_{v_{2d-1},..,v_{3}}(v_{1}))$$

We have

$$\mu_{U(d)}(du)=\mu_{S^{1}}(dv_{1}),....,\mu_{S^{2d-3}}(dv_{2d-3})\mu_{S^{2d-1}}(dv_{2d-1})$$

Construction of unitary t-design

Conjecture 1:

Assume that the following subsets are spherical 2t-designs

$$V_{2d-1} \subset S^{2d-1}, V_{2d-3} \subset S^{2d-3}, \ldots, V_3 \subset S^3, V_1 \subset S^1$$

Then, $\{u(v_{2d-1}, .., v_1)\} \subset U(d)$ is a unitary t-design.

This conjecture is converted to the following conjecture.

Construction of unitary t-design

We choose $(j_{1,k},...., j_{t,k})$ for k=1,2,3,4.

We define d disjoint sets $J_{1,k},..,J_{d,k} \subset \{1,..,t\}$ for k=1,2,3,4 as $J_{l,k} := \{s \mid j_{s,k} = l\}$

$$V_{2d-1} \subset S^{2d-1}, V_{2d-3} \subset S^{2d-3}, ..., V_3 \subset S^3, V_1 \subset S^1$$

We inductively define the function for l=1,..,d

$$g_1(v_{2d-1}, v_{2d-3}, ..., v_1) := 1$$

$$g_{l+1}(v_{2d-1}, v_{2d-3}, ..., v_{2l+1}) := \int \mu_{S^{2l-1}}(dv_{2l-1}) g_l(v_{2d-1}, v_{2d-3}, ..., v_{2l+1}, v_{2l-1}) \cdot \prod_{s \in J_{l,1}} \left\langle j_{s,2} \middle| f_{v_{2d-1}, v_{2d-3}, ..., v_{2l+1}}(v_{2l-1}) \right\rangle \cdot \prod_{s \in J_{l,3}} \overline{\left\langle j_{s,4} \middle| f_{v_{2d-1}, v_{2d-3}, ..., v_{2l+1}}(v_{2l-1}) \right\rangle}$$

Construction of unitary t-design

Conjecture 2:

Function g_{l+1} is a polynomial with degree

$$\sum_{k=1}^{l} |J_{k,1}| + |J_{k,3}|$$

Theorem

Conjecture 2⇒Conjecture 1

Conclusion

- We have proposed a conjecture for a construction of unitary t-design.
- This application is useful in quantum information theory.

References

[1] A. Ambainis and J. Emerson, Quantum t-designs: t-wise independence in the quantum world (2007), arXiv:quant-

ph/0701126.

[2] A. Hayashi, T. Hashimoto, M. Horibe, "Reexamination of optimal quantum state estimation of pure states". Physical Review A. 72 (3): 032325 (2005).

[3] M. Hayashi, Group Representation for Quantum Theory (Springer, 2017).

[4] E. Bannai and E. Bannai, A survey on spherical designs and algebraic combinatorics on spheres, European Journal of Combinatorics 30, 1392 (2009).

[5] Y. Nakata, D. Zhao, T. Okuda, E. Bannai, Y. Suzuki, S. Tamiya, K. Heya, Z. Yan, K. Zuo, S. Tamate, Y. Tabuchi, Y. Nakamura, Quantum circuits for exact unitary t-designs and applications to higher-order randomized benchmarking

arXiv:2102.12617

[6] M. Hayashi, Y. Ouyang, Tight Cramér-Rao type bounds for multiparameter quantum metrology through conic programming, arXiv: 2209.05218