Pseudo standard entanglement structure cannot be distinguished from standard entanglement structure

Hayato Arai [1] and Masahito Hayashi[2,1]

[1] Graduate School of Mathematics, Nagoya University
[2] Shenzhen Institute for Quantum Science and Engineering, Southern University of Science and Technology

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Brief Summary

- Entanglement structures (ES)
  - is a possible structure of quantum composite system in General Probabilistic Theories (GPTs)
  - is not uniquely determined even if local structures are equivalent to the standard quantum theory.
- Pseudo standard entanglement structure (PSES) :
  - available for small error verification of all maximally entangled states
  - self-dual (= saturated situation of projectivity)
- Problem : Is there any possibility of PSES except for the standard entanglement structure?
  - infinitely many!
  - even if the PSES attains perfect discrimination of non-orthogonal states
- Variety of ES with group symmetry
  - global unitary symmetry determines ES as the standard entanglement structure (SES)

Outline

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1. Preliminary:Brief Introduction to GPTs and Entanglement Structures
2. Our Setting and Results
3. Summary and open problems

Preliminary:Definition of Quantum Theory

Model of standard quantum theory on composite system $\mathcal{H}$

- State ρ (density matrix)
  - $\rho \in \mathcal{T}_+(\mathcal{H})$
  - $\operatorname{Tr} \rho = 1$
- Measurement $\{M_i\}_i$ (POVM)
  - $M_i \in \mathcal{T}_+(\mathcal{H})$

    $\Leftrightarrow \operatorname{Tr} \rho M_i \geq 0 \ (\forall \rho \in \mathcal{T}_+(\mathcal{H}))$
  - $\sum_i M_i = I$
- Probability to get a measurement outcome
  - outcome i is measured with probability $\operatorname{Tr} \rho M_i$

$\rightarrow$ the model of quantum theory is defined by $\mathcal{T}_+(\mathcal{H})$

Assumption and Notation

$\dim(\mathcal{H}) = d$

$\mathcal{T}(\mathcal{H})$: set of Hermitian matrices on $\mathcal{H}$

$\mathcal{T}_+(\mathcal{H})$: set of positive semi-definite matrices on $\mathcal{H}$

Preliminary:Definition of the Model of Quantum Theory

Model of standard quantum theory on composite system $\mathcal{H}$

- State ρ (density matrix)
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- Probability to get a measurement outcome
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$\rightarrow$ the model of quantum theory is defined by $\mathcal{T}_+(\mathcal{H})$

$\rightarrow$

Essential Points

- $\mathrm{Tr}\,\rho M_i \geq 0$
- $\sum_i \mathrm{Tr}\,\rho M_i = 1$

$\rightarrow$ $\{\mathrm{Tr}\,\rho M_i\}_i$ is probability distribution

Preliminary:Definition of the Model of GPTs

Model of GPT with positive cone $\mathcal{K}(\subset \mathcal{T}(\mathcal{H}))$

- State ρ
  - $\rho \in \mathcal{K}$
  - $\mathrm{Tr}\,\rho = 1$
- Measurement $\{M_i\}_i$
  - $M_i \in \mathcal{K}^*$

  $\Leftrightarrow$ $\mathrm{Tr}\,\rho M_i \geq 0 \ (\forall \rho \in \mathcal{K})$
  - $\sum_i M_i = I$
- Probability to get a measurement outcome
  - outcome i is measured with probability $\mathrm{Tr}\,\rho M_i$

$\rightarrow$ a model of GPTs is defined by $\mathcal{K}$

$\rightarrow$

Essential Points

- $\mathrm{Tr}\,\rho M_i \geq 0$
- $\sum_i \mathrm{Tr}\,\rho M_i = 1$

$\rightarrow$ $\{\mathrm{Tr}\,\rho M_i\}_i$ is probability distribution

Remark

Actually, GPT deals with more general models

Entanglement Structure

Notations

- $\mathcal{T}_+(\mathcal{H}_A) \otimes \mathcal{T}_+(\mathcal{H}_B) := \{\sum_i \rho_i^A \otimes \rho_i^B \mid \rho_i^A \in \mathcal{T}_+(\mathcal{H}), \rho_i^B \in \mathcal{T}_+(\mathcal{H})\}$
- $\mathcal{K}^* := \{y \in \mathcal{T}(\mathcal{H}) \mid \mathrm{Tr}\, xy \geq 0 \ \forall x \in \mathcal{K}\}$

Entanglement Structure (ES)

- ES is a possible structure of the model of the (bipartite) composite system of quantum subsystems in GPTs
- ES is not uniquely determined to the Standard Entanglement Structure (SES) corresponding to $\mathcal{T}_+(\mathcal{H}_A \otimes \mathcal{H}_B)$
- $\mathcal{T}_+(\mathcal{H}_A) \otimes \mathcal{T}_+(\mathcal{H}_B) \subset \mathcal{K} \subset (\mathcal{T}_+(\mathcal{H}_A) \otimes \mathcal{T}_+(\mathcal{H}_B))^*$
  - Local Tomography
  - No-Signaling Principle
  - Availability of separable states and separable effects
- it is important problem to derive the SES from ESs
- This talk shows the existence of “pseudo structures” of the SES.

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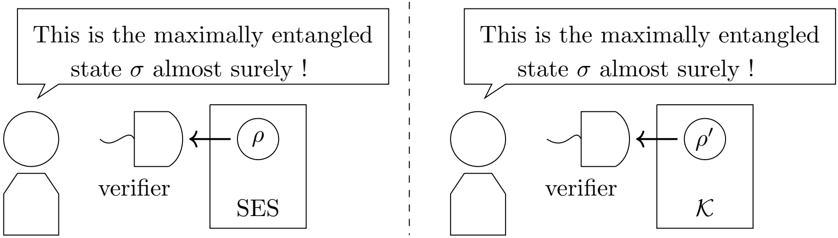
Pseudo Structures

Pseudo Standard Entanglement Structures (PSESs)
(strict definition is explained later)

- undistinguishability
  - availability of verification of any maximally entangled states with small errors
  - an experimental verification with errors does not guarantee that the prepared state is exactly the same as the maximally entangled state
- self-duality ($\Leftrightarrow \mathcal{K} = \mathcal{K}^*$)
  - saturated situation of pre-duality ($\Leftrightarrow \mathcal{K} \supset \mathcal{K}^*$)
  - pre-duality $\Leftarrow$ availability of projective measurements

- Assumption : Hereinafter, we assume $\dim \mathcal{H}_A = \dim \mathcal{H}_B = d$

Verification of maximally entangled states and ϵ-undistinguishability

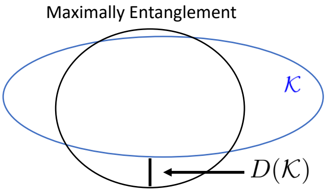


- In order to prepare a maximally entangled state σ
  - we prepare a state ρ (ρ')
  - verify that ρ (ρ') is equivalent to σ with error (probability) less than ϵ
- This task is successful if there exists such a state ρ, ρ' in a model.
  - If $\mathcal{K}$ has a state sufficiently near any maximally entangled state, $\mathcal{K}$ success this task
  - e.g. $\mathcal{K} \supset \mathcal{T}_+(\mathcal{H}_A \otimes \mathcal{H}_B)$ success this task

Verification of maximally entangled states and ϵ-undistinguishability

Because the error probability of verification (hypothesis testing) is estimated by trace norm $\|\cdot\|_1$, we introduce the following value for $\mathcal{K}$

$$D(\mathcal{K}) := \max_{\sigma \in \mathrm{ME}} \min_{\rho \in \mathcal{K}} \|\sigma - \rho\|_1$$



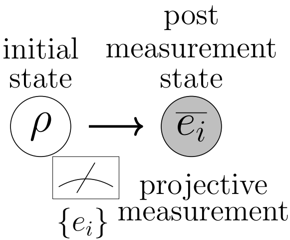
- if $\mathcal{K}$ contains all maximally entangled states, then $D(\mathcal{K}) = 0$
- ϵ-undistinguishability
  - $D(\mathcal{K}) \leq \epsilon$
  - if $\mathcal{K}$ is ϵ-undistinguishable, we cannot deny the possibility of $\mathcal{K}$ by any verification of maximally entangled states with larger errors than ϵ

Relation between projectivity and pre-duality

- Projectivity
  - For any (pure) effect e, there exists projective measurement $\{e_i\}$ such that an element e_{i_0} is equal to e, and the post-measurement state is given as $\overline{e_{i_0}} := e_{i_0} / \operatorname{Tr} e_{i_0}$.

- $\mathcal{K}^*$ is generated by (pure) effects
- there exists a correspondence $e_i \mapsto \overline{e_{i_0}}$
- because $\overline{e_{i_0}}$ is a state, $\overline{e_{i_0}} \in \mathcal{K}$

→ Projectivity derives pre-duality, i.e., $\mathcal{K} \supset \mathcal{K}^*$



- With preserving projectivity, state space is restricted and effect space is extended, then we obtain self-dual cone
  - (actually, this is not trivial and we show this statement later)

The definition of ϵ-PSESs

- ϵ-PSES is an ES with
  - ϵ-undistinguishability
  - self-duality

Definition (ϵ-Pseudo Standard Entanglement Structure)

If an ES $\mathcal{K}$ satisfiesϵ-undistinguishability and self-duality, we say that $\mathcal{K}$ is an ϵ-PSES.

- SES is an example of ϵ-PSES for $\epsilon \geq 0$
- No other simple example is not found
  - if $\mathcal{K} \supsetneq \mathrm{SES}$, $\mathcal{K}$ is not self-dual
  - if $\mathcal{K} \subsetneq \mathrm{SES}$, $\mathcal{K}$ is not ϵ-undistinguishable for small ϵ.

Q. Is there any other example of ϵ-PSES for small ϵ ? $\rightarrow$ Yes!

Main Theorem

Given $\epsilon > 0$, there exists exactly different infinite models of ϵ-PSES.

Self-dual modification

Main Theorem 1

Given any pre-dual cone $\mathcal{K}$, there exists a self-dual cone $\tilde{\mathcal{K}}$ such that $\mathcal{K} \supset \tilde{\mathcal{K}} \supset \mathcal{K}^*$.

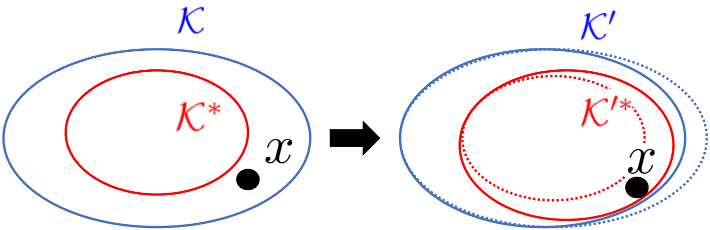
- Due to this theorem, self-duality is a natural consequence of projectivity.
- For proofs, we applying Zorn's Lemma for a certain ordered set ($\rightarrow$The proof is not constructive)
- Different pre-dual cone cannot always be modified to different self-dual cone (This problem is solved by following theorem)

Main Theorem 2

Given an exact hierarchy of pre-dual cones $\mathcal{K}_1 \supsetneq \cdots \supsetneq \mathcal{K}_n$, there exist exactly different self-dual cones $\tilde{\mathcal{K}}_i$ such that $\mathcal{K}_i \supset \tilde{\mathcal{K}}_i \supset \mathcal{K}_i^*$.

- pre-dual cones $\mathcal{K}_1 \supsetneq \cdots \supsetneq \mathcal{K}_n \Rightarrow$ exactly different self-dual cones $\tilde{\mathcal{K}}_i$

Idea of Proof of Theorem 1



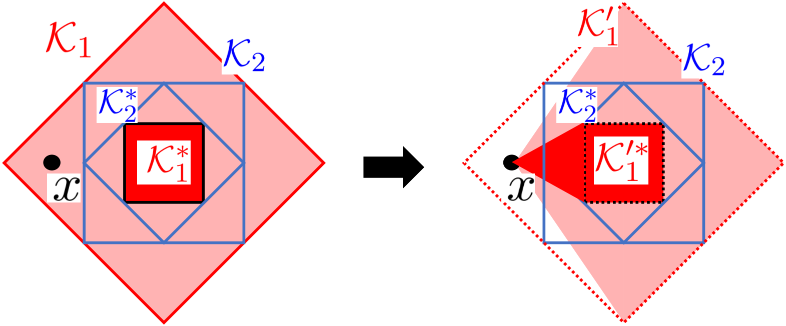
Remark
$\mathcal{K} \supset \mathcal{K}' \Leftrightarrow \mathcal{K}^* \subset \mathcal{K}'^*$

- if $\mathcal{K} \subsetneq \mathcal{K}^*$, there exists $x \in \mathcal{K} \setminus \mathcal{K}^*$
- $\mathcal{K}'^* := \mathcal{K}^* + x$, $\mathcal{K}' = (\mathcal{K}'^*)^*$
- then, x belongs to both $\mathcal{K}'$ and $\mathcal{K}'^*$

$\rightarrow$ $\mathcal{K} \subsetneq \mathcal{K}' \subset \mathcal{K}'^* \subsetneq \mathcal{K}^*$

- repeat such step infinitely many times (by Zron's Lemma)

Idea of Proof of Theorem 2



- $\mathcal{K}_1 \supsetneq \mathcal{K}_2 \supsetneq \mathcal{K}_2^* \supsetneq \mathcal{K}_1^*$ $\rightarrow$ there exists $x \in \mathcal{K}_1 \setminus \mathcal{K}_2$
- $\mathcal{K}_1'^* := \mathcal{K}_1^* + x$ $\quad \rightarrow \mathcal{K}_1'^* \not\subset \mathcal{K}_2$

$\rightarrow$ $\mathcal{K}_1'^* \subset \tilde{\mathcal{K}}_1' \not\subset \tilde{\mathcal{K}}_2$ ($\because \tilde{\mathcal{K}}_2 \subset \mathcal{K}_2$)

$\rightarrow$ $\tilde{\mathcal{K}}_1' \neq \tilde{\mathcal{K}}_2$

The existence of PSESs

- In order to apply main theorem 2, we construct an exact hierarchy of pre-dual cones
- in this paper, we construct pre-dual cone $\mathcal{K}_r$ for a parameter $r > 0$
- $\mathcal{K}_r \supsetneq \mathcal{K}_{r'}$ for $r' < r \leq r_0 \quad \rightarrow \quad \{\mathcal{K}_r\}$ is an exact hierarchy

$\rightarrow$ Main Theorem 2 implies that $\tilde{\mathcal{K}}_r$ with each r is exactly different

- $D(\tilde{\mathcal{K}}_r) \leq D(\mathcal{K}_r^*) \leq \epsilon$ for sufficiently small r.

$\rightarrow$ $\tilde{\mathcal{K}}_r$ is a ϵ-PSES.

Main Theorem 3

Given $\epsilon > 0$, there exists exactly different infinite models of ϵ-PSES.

Other results

- an ϵ-PSES has extraordinary performance for perfect discrimination
  - an ϵ-PSES has non-orthogonal perfectly distinguishable states
  - In the SES, orthogonale $\Leftrightarrow$ perfectly distinguishable

Main Theorem 4

For any $\epsilon > 0$, there is an ϵ-PSES that contains a measurement discriminating two non-orthogonal states perfectly.

- Group symmetric condition characterize the SES uniquely
  - G-symmetric cone $\mathcal{K}$: $g(x) \in \mathcal{K}$ for any $x \in \mathcal{K}$ and any $g \in G$.

$$\mathrm{GU}(A;B) := \{g \in \mathrm{GL}(\mathcal{T}(\mathcal{H}_A \otimes \mathcal{H}_B)) \mid g(\cdot) := U^\dagger(\cdot)U, \quad U \text{ is a unitary matrix on } \mathcal{H}_A \otimes \mathcal{H}_B\}.$$

Main theorem 5

For ES $\mathcal{K}$, $\mathcal{K}$ is $\mathrm{GU}(A;B)$-symmetric iff $\mathcal{K} = \mathrm{SES}$

Outline

Summary and open problems

- There are many possibilities of ESs different from the SES.
- There exists infinite examples of ϵ-PSESs
  - self-duality
  - ϵ-undistinguishability

$\rightarrow$ There exists another possibility of ESs that cannot be denied by verification of maximally entangled state with errors

OPEN. Explicit construction of ϵ-PSESs

- Some ϵ-PSES can discriminate non-orthogonal states

$\rightarrow$ Verification of maximally entanglement state with errors cannot deny the possibility of an ES with such extraordinary performance for state discrimination

OPEN. Any ϵ-PSES can discriminate non-orthogonal states except for the SES?

- $\mathrm{GU}(A;B)$-symmetry uniquely determines the SES

OPEN. there exists $\mathrm{LU}(A;B)$-symmetric ϵ-PSES except for the SES ?