New additivity properties of the relative entropy of entanglement and its generalizations

Roberto Rubboli, Marco Tomamichel

September 29, 2022

Outline

1. Entangled monotones
   - The additivity question

Outline

Outline

Outline

1. Entanglement monotones
   - The additivity question

2. Additivity of the relative entropy of entanglement
   - First class
   - Second class

3. Generalizations of the relative entropy of entanglement
   - α-z Rényi relative entropy of entanglement
   - Additivity of the α-z Rényi relative entropy of entanglement
   - Applications to catalytic transformations of pure entangled states

4. Non-Additivity of the entanglement monotones based on a quantum relative entropy

Entanglement monotones

Entanglement monotones

- A state N-partite state σ is separable (SEP) if

$$\sigma = \sum_i p_i \sigma_i^1 \otimes \ldots \otimes \sigma_i^N \tag{1}$$

for local states $\sigma_i^j \in \mathcal{S}(A_j)$ and a probability distribution $\{p_i\}$. Otherwise it is entangled.

Entanglement monotones

- A state N-partite state σ is separable (SEP) if

$$\sigma = \sum_i p_i \sigma_i^1 \otimes \ldots \otimes \sigma_i^N \tag{1}$$

for local states $\sigma_i^j \in \mathcal{S}(A_j)$ and a probability distribution $\{p_i\}$. Otherwise it is entangled.

- The function $\mathfrak{R} : \mathcal{S}(A) \to [0, +\infty]$ is an entanglement monotone if it does not increase under any local operations and classical communication (LOCC), i.e., if

$$\mathfrak{R}(\rho) \geq \mathfrak{R}(\mathcal{E}(\rho)) \tag{2}$$

for any state ρ and LOCC operation $\mathcal{E}$.

Examples: relative entropy of entanglement, generalized robustness of entanglement, log-negativity...

The additivity question

A fundamental problem is to establish whether a certain entanglement monotone is tensor-additive.
We say that $\mathfrak{R}$ is *tensor-additive* for the states ρ_1 and ρ_2 if

$$\mathfrak{R}(\rho_1 \otimes \rho_2) = \mathfrak{R}(\rho_1) + \mathfrak{R}(\rho_2) \tag{3}$$

[1] Vollbrecht, Karl Gerd H., and Reinhard F. Werner. "Entanglement measures under symmetry." Physical Review A 64, no. 6 (2001): 062307.

[2] Zhu, Huangjun, Lin Chen, and Masahito Hayashi. "Additivity and non-additivity of multipartite entanglement measures." New Journal of Physics 12, no. 8 (2010): 083002.

[3] Zhu, Huangjun, Masahito Hayashi, and Lin Chen. "Coherence and entanglement measures based on Rényi relative entropies." Journal of Physics A: Mathematical and Theoretical 50, no. 47 (2017): 475303.

The additivity question

A fundamental problem is to establish whether a certain entanglement monotone is tensor-additive.
We say that $\mathfrak{R}$ is *tensor-additive* for the states ρ_1 and ρ_2 if

$$\mathfrak{R}(\rho_1 \otimes \rho_2) = \mathfrak{R}(\rho_1) + \mathfrak{R}(\rho_2) \tag{3}$$

- Is $\mathfrak{R}$ additive for any states ρ_1 and ρ_2? [1] [2]

[1] Vollbrecht, Karl Gerd H., and Reinhard F. Werner. "Entanglement measures under symmetry." Physical Review A 64, no. 6 (2001): 062307.

[2] Zhu, Huangjun, Lin Chen, and Masahito Hayashi. "Additivity and non-additivity of multipartite entanglement measures." New Journal of Physics 12, no. 8 (2010): 083002.

[3] Zhu, Huangjun, Masahito Hayashi, and Lin Chen. "Coherence and entanglement measures based on Rényi relative entropies." Journal of Physics A: Mathematical and Theoretical 50, no. 47 (2017): 475303.

The additivity question

A fundamental problem is to establish whether a certain entanglement monotone is tensor-additive.
We say that $\mathfrak{R}$ is *tensor-additive* for the states ρ_1 and ρ_2 if

$$\mathfrak{R}(\rho_1 \otimes \rho_2) = \mathfrak{R}(\rho_1) + \mathfrak{R}(\rho_2) \tag{3}$$

- Is $\mathfrak{R}$ additive for any states ρ_1 and ρ_2? [1] [2]
- What are the minimum requirement on the states ρ_1 and ρ_2 to ensure additivity?[3]

[1] Vollbrecht, Karl Gerd H., and Reinhard F. Werner. "Entanglement measures under symmetry." Physical Review A 64, no. 6 (2001): 062307.

[2] Zhu, Huangjun, Lin Chen, and Masahito Hayashi. "Additivity and non-additivity of multipartite entanglement measures." New Journal of Physics 12, no. 8 (2010): 083002.

[3] Zhu, Huangjun, Masahito Hayashi, and Lin Chen. "Coherence and entanglement measures based on Rényi relative entropies." Journal of Physics A: Mathematical and Theoretical 50, no. 47 (2017): 475303.

Additivity properties of the relative entropy of entanglement

Additivity of the relative entropy of entanglement

Let $\rho, \sigma \in \mathcal{S}(A)$. The relative entropy is

$$D(\rho\|\sigma) = \begin{cases} \mathrm{Tr}[\rho(\log\rho - \log\sigma)] & \text{if } \mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma) \\ +\infty & \text{else} \end{cases} \tag{4}$$

Additivity of the relative entropy of entanglement

Let $\rho, \sigma \in \mathcal{S}(A)$. The relative entropy is

$$D(\rho\|\sigma) = \begin{cases} \mathrm{Tr}[\rho(\log\rho - \log\sigma)] & \text{if } \mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma) \\ +\infty & \text{else} \end{cases} \tag{4}$$

The relative entropy of entanglement is

$$\mathfrak{D}(\rho) := \min_{\sigma \in \mathrm{SEP}} D(\rho\|\sigma) \tag{5}$$

Additivity of the relative entropy of entanglement

Let $\rho, \sigma \in \mathcal{S}(A)$. The relative entropy is

$$D(\rho\|\sigma) = \begin{cases} \mathrm{Tr}[\rho(\log\rho - \log\sigma)] & \text{if } \mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma) \\ +\infty & \text{else} \end{cases} \tag{4}$$

The relative entropy of entanglement is

$$\mathfrak{D}(\rho) := \min_{\sigma \in \mathrm{SEP}} D(\rho\|\sigma) \tag{5}$$

- A *maximally correlated state* has the form

$$\rho_{\mathrm{MC}} = \sum_{jk} \rho_{jk} |j,j\rangle\langle k,k| \,. \tag{6}$$

A bipartite pure state $\rho = \sum_{jk} \sqrt{p_j p_k} |j,j\rangle\langle k,k|$ is a maximally correlated state.

Theorem 1

Let ρ_1 be a maximally correlated state. Then, for any state ρ_2 we have

$$\mathfrak{D}(\rho_1 \otimes \rho_2) = \mathfrak{D}(\rho_1) + \mathfrak{D}(\rho_2) \,. \tag{7}$$

Additivity of the relative entropy of entanglement

Theorem 2

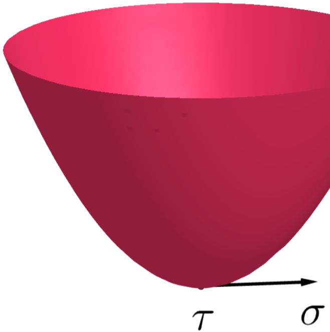
Let ρ_1 be a N-partite state. Moreover, let $\tau_1 \in \arg\min_{\sigma \in \mathsf{SEP}} D(\rho_1 \| \sigma)$. If $[\rho_1, \tau_1] = 0$ and $\rho_1 \tau_1^{-1}$ is non-negative, then for any N-partite state ρ_2 we have that

$$\mathfrak{D}(\rho_1 \otimes \rho_2) = \mathfrak{D}(\rho_1) + \mathfrak{D}(\rho_2) \,. \tag{8}$$

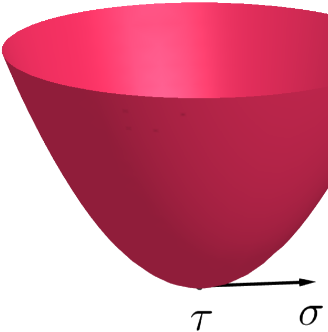
This class includes the separable, Bell diagonal, generalized Dicke and the isotropic states.

For some tasks (see later) we are interested in other resource monotones. What about the additivity properties of the other resource monotones?

Main idea



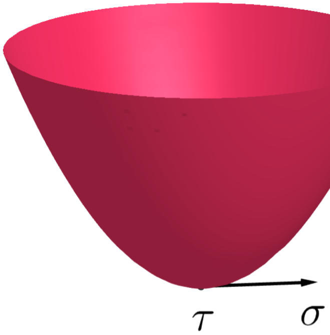
Main idea



Let ρ be a quantum state. Then $\tau \in \arg\min_{\sigma \in \mathsf{SEP}} D(\rho\|\sigma)$ if and only if $\mathrm{Tr}(\sigma\, \Xi(\rho, \tau)) \leq 1$ for all $\sigma \in$ SEP where

$$\Xi(\rho, \tau) = \int_0^\infty (\tau + t)^{-1} \rho (\tau + t)^{-1} \mathrm{d}t \tag{9}$$

Main idea



Let ρ be a quantum state. Then $\tau \in \arg\min_{\sigma \in \mathsf{SEP}} D(\rho\|\sigma)$ if and only if $\mathrm{Tr}(\sigma\, \Xi(\rho,\tau)) \leq 1$ for all $\sigma \in$ SEP where

$$\Xi(\rho,\tau) = \int_0^\infty (\tau+t)^{-1}\rho(\tau+t)^{-1}\mathrm{d}t \tag{9}$$

$\rightarrow$ We want to show that

$$\mathrm{Tr}(\sigma\Xi(\rho_1\otimes\rho_2, \tau_1\otimes\tau_2)) \leq 1 \ \forall\sigma \in \mathsf{SEP}\,, \tag{10}$$

if $\mathrm{Tr}(\sigma\Xi(\rho_i,\tau_i)) \leq 1 \ \forall\sigma \in \mathsf{SEP} \quad i = 1,2\,.$ + additional constraints on ρ_1 and τ_1.

Generalizations

α-z Rényi relative entropy of entanglement

The α-z Rényi relative entropy provides a general framework to address different families of quantum Rényi divergences.

Let $\alpha \in (0,1) \cup (1,\infty)$, $z > 0$ and $\rho, \sigma \in \mathcal{S}(A)$ with $\rho \neq 0$. Then the α-z *Rényi relative entropy* of σ with ρ is defined as [4]

$$D_{\alpha,z}(\rho\|\sigma) := \begin{cases} \frac{1}{\alpha-1} \log \mathrm{Tr}\left(\rho^{\frac{\alpha}{2z}} \sigma^{\frac{1-\alpha}{z}} \rho^{\frac{\alpha}{2z}}\right)^z & \text{if } (\alpha < 1 \wedge \rho \not\perp \sigma) \vee \rho \ll \sigma \\ +\infty & \text{else} \end{cases} . \quad (11)$$

[4] Audenaert, Koenraad MR, and Nilanjana Datta. 'α-z-Rényi relative entropies.' Journal of Mathematical Physics 56, no. 2 (2015): 022202.

α-z Rényi relative entropy of entanglement

The α-z Rényi relative entropy provides a general framework to address different families of quantum Rényi divergences.

Let $\alpha \in (0,1) \cup (1,\infty)$, $z > 0$ and $\rho, \sigma \in \mathcal{S}(A)$ with $\rho \neq 0$. Then the α-z *Rényi relative entropy* of σ with ρ is defined as [4]

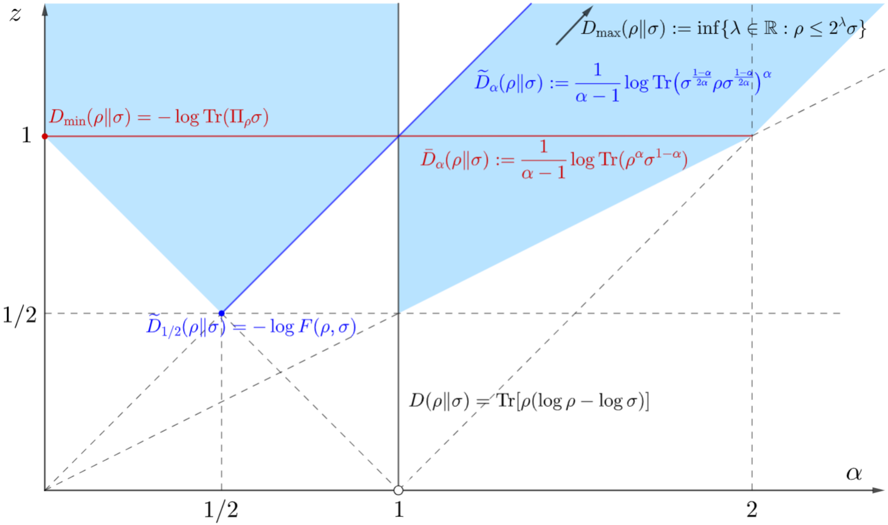
$$D_{\alpha,z}(\rho\|\sigma) := \begin{cases} \frac{1}{\alpha-1} \log \mathrm{Tr}\left(\rho^{\frac{\alpha}{2z}} \sigma^{\frac{1-\alpha}{z}} \rho^{\frac{\alpha}{2z}}\right)^z & \text{if } (\alpha < 1 \wedge \rho \not\perp \sigma) \vee \rho \ll \sigma \\ +\infty & \text{else} \end{cases} . \quad (11)$$

The α-z *Rényi relative entropy of entanglement* is

$$\mathfrak{D}_{\alpha,z}(\rho) := \min_{\sigma \in \mathrm{SEP}} D_{\alpha,z}(\rho\|\sigma) . \quad (12)$$

[4] Audenaert, Koenraad MR, and Nilanjana Datta. 'α-z-Rényi relative entropies.' Journal of Mathematical Physics 56, no. 2 (2015): 022202.

α-z Rényi relative entropy



Some well-known points

Some well-known points

- $\alpha = 1$ $(z \neq 0)$: **Relative entropy of entanglement**

Some well-known points

- $\alpha = 1$ $(z \neq 0)$: **Relative entropy of entanglement**
- $\alpha = z = \infty$: **Generalized (log) robustness of entanglement**
  The generalized robustness is given by

$$\mathfrak{R}_g(\rho) := \min \left\{ s \geq 0 : \exists \omega \in \mathcal{S}(A) \text{ s.t } \frac{1}{1+s} (\rho + s\omega) \in \mathsf{SEP} \right\} . \tag{13}$$

  We have $\mathfrak{D}_{\infty,\infty}(\rho) = \log\left(1 + \mathfrak{R}_g(\rho)\right)$.

Some well-known points

- $\alpha = 1$ $(z \neq 0)$: **Relative entropy of entanglement**
- $\alpha = z = \infty$: **Generalized (log) robustness of entanglement**
  The generalized robustness is given by

$$\mathfrak{R}_g(\rho) := \min\left\{ s \geq 0 : \exists \omega \in \mathcal{S}(A) \text{ s.t } \frac{1}{1+s}(\rho + s\omega) \in \mathsf{SEP} \right\}. \quad (13)$$

  We have $\mathfrak{D}_{\infty,\infty}(\rho) = \log(1 + \mathfrak{R}_g(\rho))$.
- $\alpha = z = 1/2$: **Geometric measure of entanglement**

$$E_G(|\psi\rangle) = 1 - \max_{|\phi\rangle \in \mathsf{PRO}} |\langle \phi|\psi\rangle|^2 \quad (14)$$

$$E_G(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E_G(|\psi_i\rangle). \quad (15)$$

  We have $\mathfrak{D}_{1/2,1/2}(\rho) = -\log F_s(\rho) = -\log(1 - E_G(\rho))$ where $F_s(\rho) = \max_{\sigma \in \mathsf{SEP}} F(\rho, \sigma)$ is the fidelity of separability.

Additivity of the α-z Rényi relative entropy of entanglement

Theorem 3

Let ρ_1 *be a maximally correlated state and* $(\alpha, z) \in \mathcal{D}$*. Then, for any state* ρ_2 *we have*

$$\mathfrak{D}_{\alpha,z}(\rho_1 \otimes \rho_2) = \mathfrak{D}_{\alpha,z}(\rho_1) + \mathfrak{D}_{\alpha,z}(\rho_2)\,. \tag{16}$$

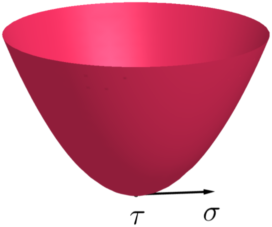
Theorem 4

Let ρ_1 *be a N-partite state and* $(\alpha, z) \in \mathcal{D}$*. Moreover, let* $\tau_1 \in \arg\min_{\sigma \in \mathsf{SEP}} D_{\alpha,z}(\rho_1 \| \sigma)$*. If* $[\rho_1, \tau_1] = 0$ *and* $\rho_1^{\alpha} \tau_1^{-\alpha}$ *is non-negative, then for any N-partite state* ρ_2 *we have that*

$$\mathfrak{D}_{\alpha,z}(\rho_1 \otimes \rho_2) = \mathfrak{D}_{\alpha,z}(\rho_1) + \mathfrak{D}_{\alpha,z}(\rho_2)\,. \tag{17}$$

α-z Rényi relative entropy

For the α-z Rényi relative entropy



Theorem 5

Let ρ be a quantum state and $(\alpha, z) \in \mathcal{D}$. Then $\tau \in \arg\min_{\sigma\in\mathsf{SEP}} D_{\alpha,z}(\rho\|\sigma)$ if and only if $\mathrm{Tr}(\sigma\,\Xi_{\alpha,z}(\rho,\tau)) \leq Q_{\alpha,z}(\rho\|\tau)$ for all $\sigma \in \mathsf{SEP}$ where

$$\Xi_{\alpha,z}(\rho,\tau) = \begin{cases} \mathrm{sinc}\left(\pi\frac{1-\alpha}{z}\right)\int_0^\infty (\tau+t)^{-1}\rho^{\frac{\alpha}{2z}}\left(\rho^{\frac{\alpha}{2z}}\tau^{\frac{1-\alpha}{z}}\rho^{\frac{\alpha}{2z}}\right)^{z-1}\rho^{\frac{\alpha}{2z}}(\tau+t)^{-1}t^{\frac{1-\alpha}{z}}\,\mathrm{d}t & \text{if } |(1-\alpha)/z| \neq 1 \\ \rho^{\frac{\alpha}{2(1-\alpha)}}\left(\rho^{\frac{\alpha}{2(1-\alpha)}}\tau\rho^{\frac{\alpha}{2(1-\alpha)}}\right)^{-\alpha}\rho^{\frac{\alpha}{2(1-\alpha)}} & \text{if } |(1-\alpha)/z| = 1 \end{cases} \tag{18}$$

and any resource theory... SEP $\rightarrow$ free set $\mathcal{F}$

Applications to catalytic transformations of pure entangled states

Understand the power of mixed catalysts

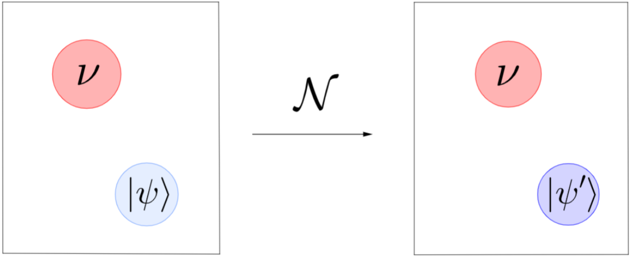
We have

$$\mathfrak{D}_{\alpha,z}(|\psi\rangle) + \mathfrak{D}_{\alpha,z}(\nu) = \mathfrak{D}_{\alpha,z}(|\psi\rangle \otimes \nu) \geq \mathfrak{D}_{\alpha,z}(|\psi'\rangle \otimes \nu) = \mathfrak{D}_{\alpha,z}(|\psi'\rangle) + \mathfrak{D}_{\alpha,z}(\nu) \tag{19}$$

This implies the following set of necessary conditions:

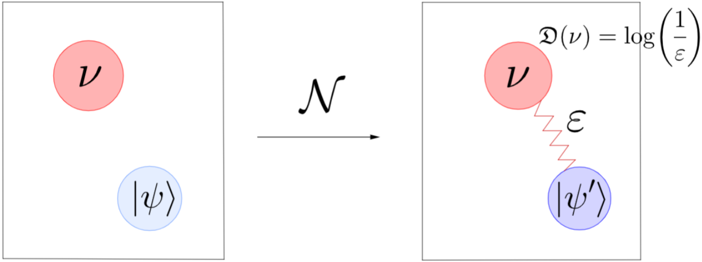
$$\mathfrak{D}_{\alpha,z}(|\psi\rangle) \geq \mathfrak{D}_{\alpha,z}(|\psi'\rangle) \qquad \forall(\alpha, z) \in \mathcal{D} \tag{20}$$

(Let $|\psi\rangle = \sum_i \sqrt{p_i}|i,i\rangle$. Then $\mathfrak{D}_{\alpha,z}(|\psi\rangle) = H_\beta(\vec{p})$ where $(1-\alpha)/z + 1/\beta = 1$.)



Find fundamental limits of correlated catalytic transformations

The additivity result for $\alpha = z \in [1/2, 1)$ is a key ingredient to prove that, for a large class of correlated catalytic transformations, the resource of the catalyst must diverge as the correlations vanish.[5]



[5] Rubboli, Roberto, and Marco Tomamichel. "Fundamental limits on correlated catalytic state transformations." Physical Review Letters 129, no. 12 (2022): 120506.

Non-Additivity of the entanglement monotones based on a quantum relative entropy

Non-additivity of entanglement monotones based on a quantum relative entropy

We define the monotone based on a quantum relative entropy

$$\min_{\sigma \in \mathsf{SEP}} \mathbb{D}(\rho \| \sigma) \tag{21}$$

Here, $\mathbb{D}$ satisfies [6]

- Data-processing inequality:
  $\mathbb{D}(\rho\|\sigma) \geq \mathbb{D}(\mathcal{E}(\rho)\|\mathcal{E}(\sigma))$ for any quantum channel $\mathcal{E}$.
- Additivity under tensor products:
  $\mathbb{D}(\rho_1 \otimes \rho_2 \| \sigma_1 \otimes \sigma_2) = \mathbb{D}(\rho_1\|\sigma_1) + \mathbb{D}(\sigma_2\|\sigma_2)$
- Normalization condition:
  $\mathbb{D}(|0\rangle\langle 0| \| \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|) = 1$

[6] Gour, Gilad, and Marco Tomamichel. "Optimal extensions of resource measures and their applications." Physical Review A 102, no. 6 (2020): 062401.

Non-additivity of entanglement monotones based on a quantum relative entropy

This monotone is not additive for general states.
For $d \gg 1$ we have that

$$\min_{\sigma \in \mathsf{SEP}} \mathbb{D}(\rho_- \otimes \rho_- \| \sigma) \sim \min_{\sigma \in \mathsf{SEP}} \mathbb{D}(\rho_- \| \sigma) \quad (22)$$

$$\neq 2 \min_{\sigma \in \mathsf{SEP}} \mathbb{D}(\rho_- \| \sigma) \quad (23)$$

(ρ_- the bipartite antisymmetric (Werner) state).
The same holds for the minimization over PPT states.

Thanks for your attention!