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A simple and tighter derivation of achievability for classical communication over quantum channels

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A General Recipe for **One-Shot** QIT

No specific structures on the state/channel

Quantum Information-Theoretic Tasks

↓

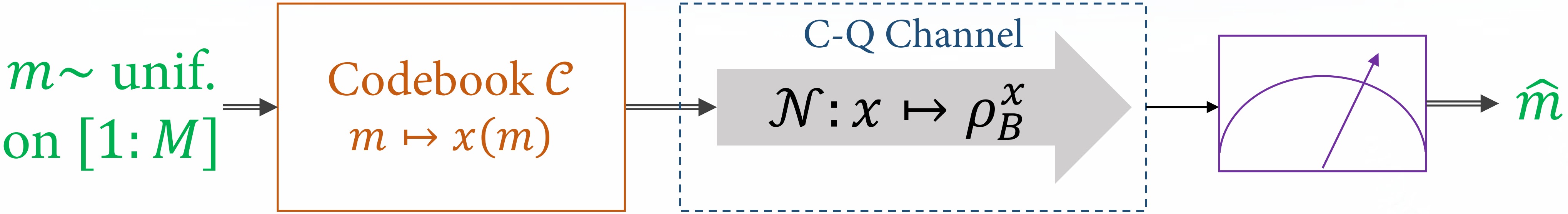
Reduction to a one-shot quantum hypothesis testing (QHT) problem

↓

One-shot characterizations of QHT and its asymptotic expansions

A Powerful Information-Spectrum Method

- Classical-quantum channel coding



- Hayashi–Nagaoka operator inequality: for $0 \leq A \leq I, B \geq 0$,

$$I - \frac{A}{A+B} \leq (1+c)(I-A) + (2+c+c^{-1})B, \qquad \forall c > 0$$

  - A noncommutative quotient:

$$\frac{A}{B} := B^{-\frac{1}{2}} A B^{-\frac{1}{2}}$$

IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. 49, NO. 7, JULY 2003 1753

General Formulas for Capacity of Classical-Quantum Channels

Masahito Hayashi and Hiroshi Nagaoka, *Member, IEEE*

A view from the Quantum Union Bound

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royalsocietypublishing.org/journal/rspa

Research

Check for updates

Union bound for quantum information processing

Samad Khabbazi Oskouei[1], Stefano Mancini[2,3] and Mark M. Wilde[4]

[1]Department of Mathematics, Islamic Azad University,

Theorem 2.1 (Quantum union bound). *Let ρ be a density operator acting on a separable Hilbert space $\mathcal{H}$, let $\{P_i\}_{i=1}^{L}$ be an arbitrary set of projectors, each acting on $\mathcal{H}$, and let $c > 0$ be an arbitrary positive constant. Then*

$$1 - \mathrm{Tr}\{P_L P_{L-1} \cdots P_1 \rho P_1 \cdots P_{L-1}\} \leq (1+c)\mathrm{Tr}\{(I - P_L)\rho\} + (2 + c + c^{-1}) \sum_{i=2}^{L-1} \mathrm{Tr}\{(I - P_i)\rho\} + (2 + c^{-1})\mathrm{Tr}\{(I - P_1)\rho\}. \quad (2.1)$$

Goal

Provide a conceptually simpler and intuitive proof by directly using the pretty-good measurement.

- Tighter one-shot bounds
- Streamlined proofs to other QIT tasks

The Proposed Method

Properties of Noncommutative Minimal (1/3)

$$A \wedge B := \mathrm{argmax}_M \{\mathrm{Tr}[M] : M \leq A, M \leq B\}$$

1. (Holevo–Helstrom Theorem).

$$A \wedge B = \frac{1}{2}(A + B - |A - B|)$$

$$\mathrm{Tr}[A \wedge B] = \inf_{0 \leq T \leq I} \{\mathrm{Tr}[A(I - T)] + \mathrm{Tr}[BT]\}$$

$\mathrm{Tr}[A \wedge B]$ is the minimum "*error*" of discrimination between A and B.

Properties of Noncommutative Minimal (1/3)

1. (Holevo–Helstrom Theorem).

ELSEVIER

Information and Control

Volume 10, Issue 3, March 1967, Pages 254-291

Information and Cont...

Detection theory and quantum mechanics

Carl W. Helstrom

A. S. Holevo, An analogue of statistical decision theory and noncommutative probability theory, *Tr. Mosk. Mat. Obs.*, 1972, Volume 26, 133–149

A. S. Holevo, Remarks on Optimal Quantum Measurements, *Probl. Peredachi Inf.*, 1974, Volume 10, Issue 4, 51–55

A. S. Kholevo, Investigations in the general theory of statistical decisions, *Trudy Mat. Inst. Steklov.*, 1976, Volume 124, 3–140

Upper bounds on the error probabilities and asymptotic error exponents in quantum multiple state discrimination

J. Math. Phys. **55**, 102201 (2014); https://doi.org/10.1063/1.4898559

Koenraad M. R. Audenaert[1,2, a)] *and* Milán Mosonyi[3,4, b)]

Properties of Noncommutative Minimal (2/3)

1. (Holevo–Helstrom Theorem). $A \wedge B = \frac{1}{2}(A + B - |A - B|)$
2. (Monotone in Loewner ordering).
$$\mathrm{Tr}[A \wedge B] \leq \mathrm{Tr}[A' \wedge B'], \forall A' \geq A, \forall B' \geq B$$
3. (Monotone under trace-preserving maps).
$$\mathrm{Tr}[A \wedge B] \leq \mathrm{Tr}[\mathcal{N}(A) \wedge \mathcal{N}(B)]$$
4. (Concavity). $(A, B) \mapsto \mathrm{Tr}[A \wedge B]$ is jointly concave
5. (Direct sum). $(A \oplus A') \wedge (B \oplus B') = (A \wedge B) \oplus (A' \wedge B')$

Properties of Noncommutative Minimal (3/3)

[Audenaert *et al.*'08]

6. (Upper bound). $\mathrm{Tr}[A \wedge B] \leq \mathrm{Tr}[A^{1-s} \wedge B^s], \forall s \in [0,1]$

7. (Lower bound). $\mathrm{Tr}[A \wedge B] \geq \mathrm{Tr}\left[A \frac{B}{A+B}\right]$

- Commuting case: $\min(a, b) \geq \frac{ab}{a+b} = (\mathrm{a}^{-1} + b^{-1})^{-1}$
- A special case: $\mathrm{Tr}[A] = \mathrm{Tr}[B]$, e.g. $A = \rho$ and $B = \sigma$

  Barnum–Knill Thm.: error using PGM is at most twice of the opt. error

$$\varepsilon^{\mathrm{PGM}} = \frac{1}{2}\mathrm{Tr}\left[\rho \frac{\sigma}{\rho+\sigma}\right] + \frac{1}{2}\mathrm{Tr}\left[\sigma \frac{\rho}{\rho+\sigma}\right] = \mathrm{Tr}\left[\rho \frac{\sigma}{\rho+\sigma}\right] \leq 2 \times \varepsilon^\star$$

$$\varepsilon^\star = \frac{1}{2}\mathrm{Tr}[\rho \wedge \sigma]$$

A One-Shot Achievability

Theorem. For any classical-quantum channel $x \mapsto \rho_B^x$, there exists an (M, ε)-code such that for all probability distributions p_X,

$$\varepsilon \leq \mathrm{Tr}[\rho_{XB} \wedge (M-1)\rho_X \otimes \rho_B].$$

Here, $\rho_{XB} = \sum_x p_X(x)|x\rangle\langle x| \otimes \rho_B^x$.

Proof

- Given any codebook $\mathcal{C} = \{x_1, x_2, \ldots, x_M\}$, use PGM: $\Pi_B^x := \frac{\rho_B^x}{\sum_{\bar{x}\in\mathcal{C}} \rho_B^{\bar{x}}}$

$x \in \mathcal{C}$

1. For transmitting each $x \in \mathcal{C}$, relate its error to '$\wedge$':

$$\mathrm{Tr}[\rho_B^x(I - \Pi_B^x)] = \mathrm{Tr}\left[\rho_B^x \frac{\sum_{\bar{x}\neq x} \rho_B^{\bar{x}}}{\rho_B^x + \sum_{\bar{x}\neq x} \rho_B^{\bar{x}}}\right] \leq \mathrm{Tr}\left[\rho_B^x \wedge \sum_{\bar{x}\neq x} \rho_B^{\bar{x}}\right]$$

2. Expectation: $\mathbb{E}_x \mathbb{E}_{\bar{x}|x} \mathrm{Tr}\left[\rho_B^x \wedge \sum_{\bar{x}\neq x} \rho_B^{\bar{x}}\right] \leq \mathbb{E}_x \mathrm{Tr}\left[\rho_B^x \wedge \sum_{\bar{x}\neq x} \mathbb{E}_{\bar{x}|x} \rho_B^{\bar{x}}\right]$

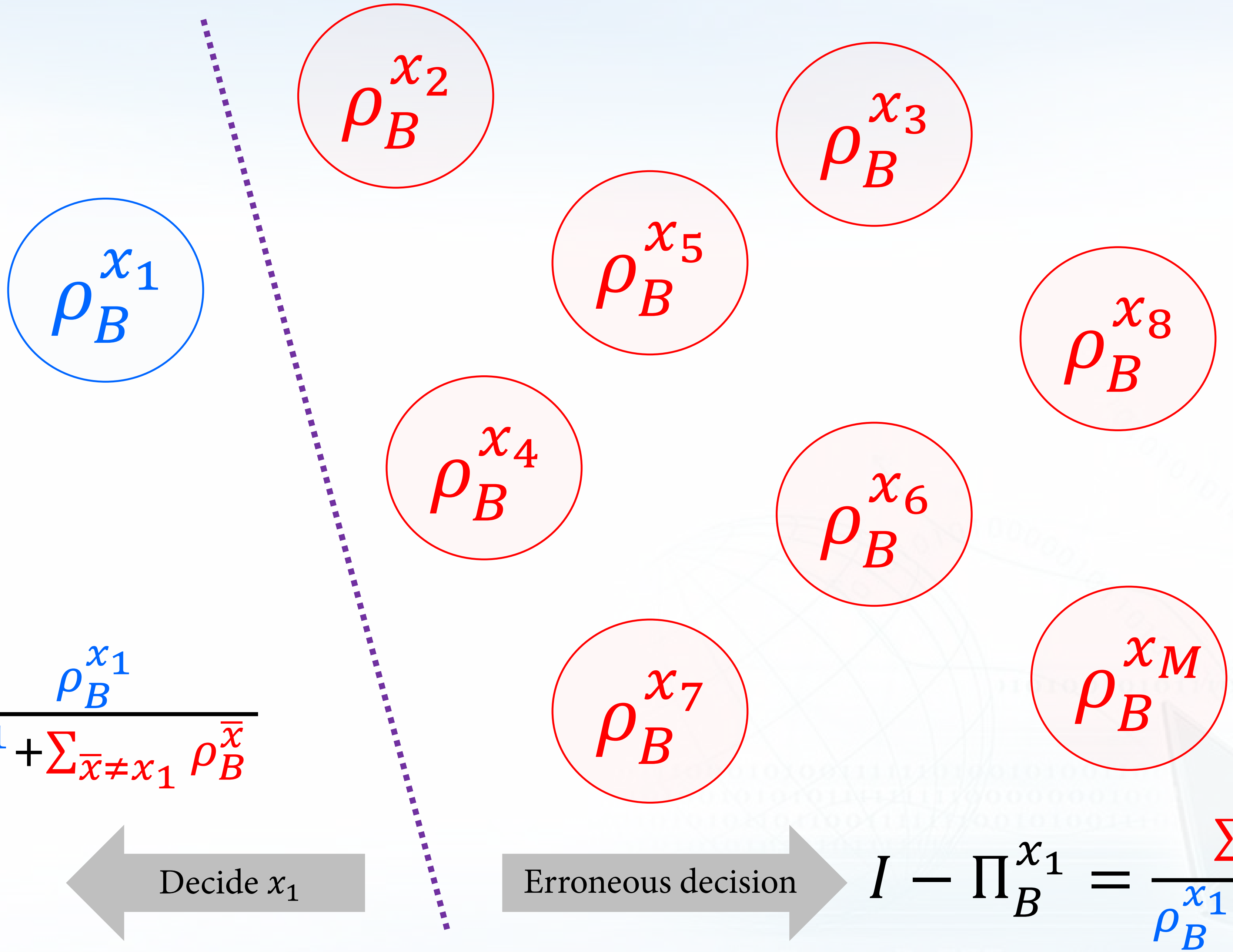
Pairwise independence of random codebook →

$$= \mathbb{E}_{x\sim p_X} \mathrm{Tr}[\rho_B^x \wedge (M-1)\rho_B]$$

$$= \mathrm{Tr}[\rho_{XB} \wedge (M-1)\rho_X \otimes \rho_B]$$

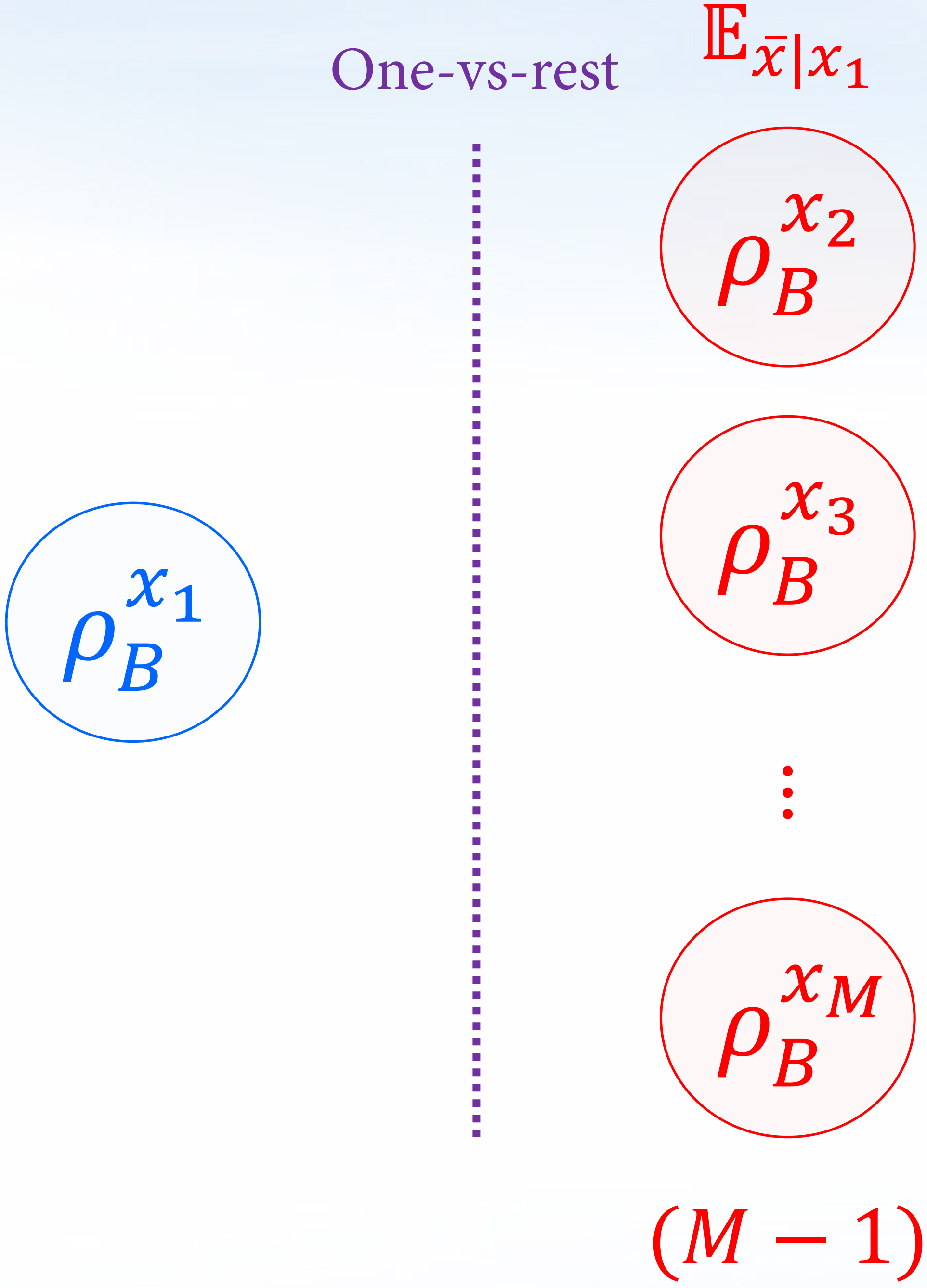
□

One-vs-rest

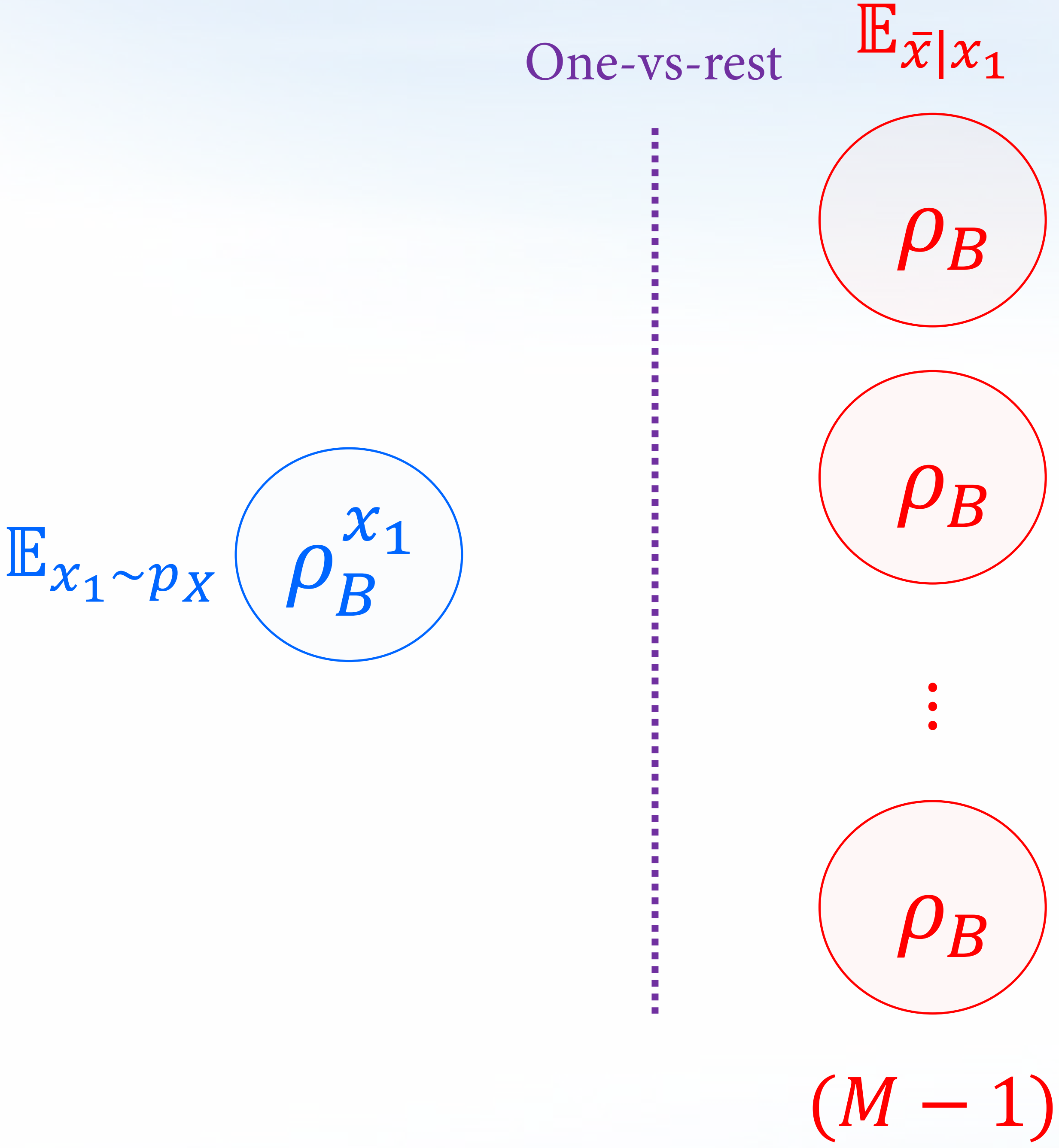


$$\Pi_B^{x_1} = \frac{\rho_B^{x_1}}{\rho_B^{x_1} + \sum_{\bar{x} \neq x_1} \rho_B^{\bar{x}}}$$

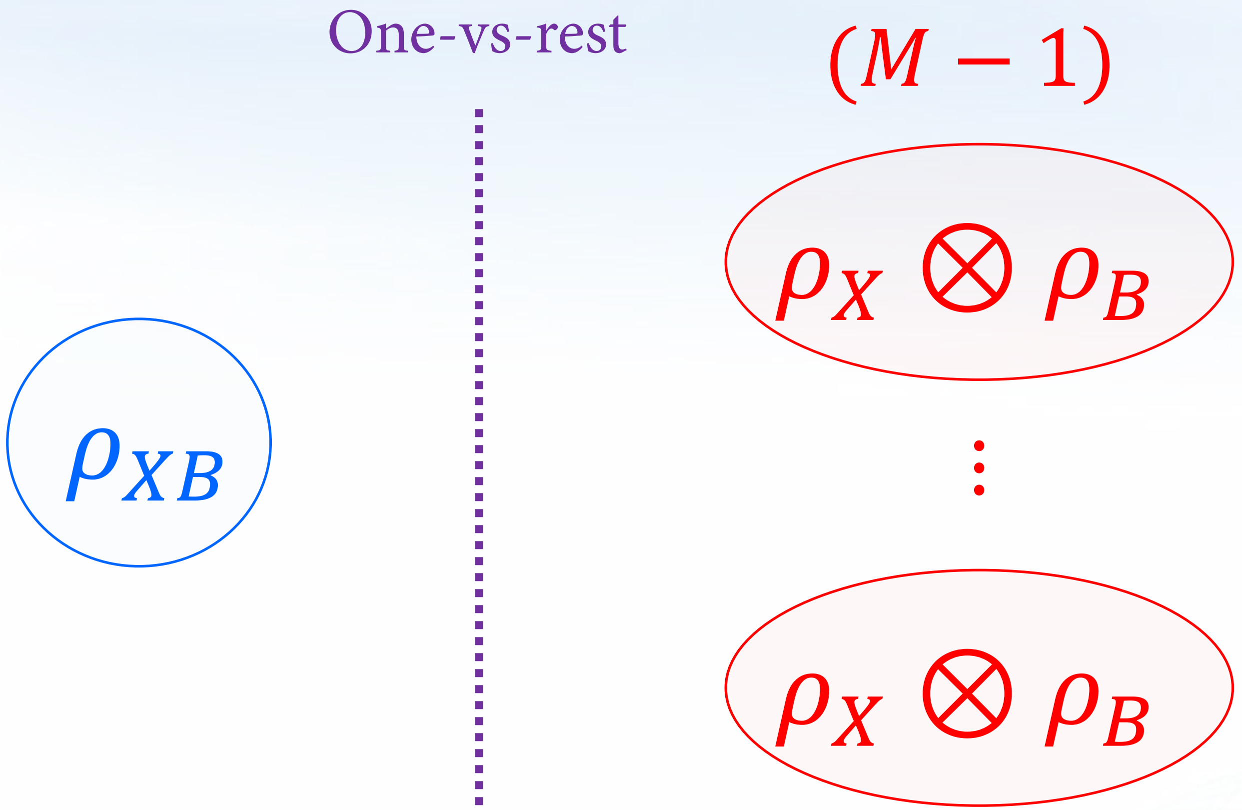
$$I - \Pi_B^{x_1} = \frac{\sum_{\bar{x} \neq x_1} \rho_B^{\bar{x}}}{\rho_B^{x_1} + \sum_{\bar{x} \neq x_1} \rho_B^{\bar{x}}}$$



$$\mathrm{Tr}\left[\rho_B^{x_1} \wedge \sum_{\bar{x}\neq x_1} \rho_B^{\bar{x}}\right]$$



$$\mathbb{E}_{x_1 \sim p_X} \mathrm{Tr}\left[\rho_B^{x_1} \wedge (M-1)\rho_B\right]$$



$$\mathrm{Tr}[\rho_{XB} \wedge (M-1)\rho_X \otimes \rho_B]$$

ε is upper bounded by the error of discriminating ρ_{XB} and $(M-1)\ \rho_X \otimes \rho_B$

Implications (1/2)

- Applying an upper bound: $\mathrm{Tr}[A \wedge B] \leq \mathrm{Tr}[A^{1-s} \wedge B^s], \forall s \in [0,1]$

$$\varepsilon \leq \mathrm{e}^{-E^{\downarrow}(\log M)}$$

  - $E^{\downarrow}(\log M) := \sup_{\alpha \in [1/2,1]} \frac{1-\alpha}{\alpha} \left(D_{2-1/\alpha}(\rho_{XB} || \rho_X \otimes \rho_B)_\rho - \log M \right)$

- Comparisons to existing one-shot exponential bounds:
  - [Burnashev–Holevo'98]: $\varepsilon \leq 2\mathrm{e}^{-E^{\downarrow}(\log M)}$ for pure-state channels
  - [Hayashi'07]: $\varepsilon \leq 4\mathrm{e}^{-E^{\downarrow}(\log M)}$ for general c-q channels

- → Moderate deviation regime : $R = I(X:B) - a_n, \varepsilon \to 0$ [Cheng–Hsieh'18]

  $(a_n \to 0, n a_n^2 \to \infty)$

Implications (2/2)

- Infimum representation: $\mathrm{Tr}[A \wedge B] = \inf_{0 \le T \le I} \{\mathrm{Tr}[A(I - T)] + \mathrm{Tr}[BT]\}$

$$M \geq D_{\mathrm{h}}^{\varepsilon-\delta}(\rho_{XB} \| \rho_X \otimes \rho_B) - \log \frac{1}{\delta} \quad \forall 0 < \delta < \varepsilon$$

  - $D_{\mathrm{h}}^{\varepsilon}(\rho \| \sigma) \coloneqq \sup_{0 \le T \le I} \{-\log \mathrm{Tr}[\sigma T] : \mathrm{Tr}[\rho T] \geq 1 - \varepsilon\}$

The tightest one-shot capacity so far

- Comparisons to existing one-shot bounds:
  - [Hayashi–Nagaoka'03], [Wang–Renner'12]: $M \geq D_{\mathrm{h}}^{\varepsilon-\delta}(\cdot \| \cdot) - \log \frac{4\varepsilon}{\delta^2}$
  - [Beigi–Gohari'14]: $M \geq D_{\mathrm{s}}^{\varepsilon-\delta}(\cdot \| \cdot) - \log \frac{1-\varepsilon}{\delta} \geq D_{\mathrm{h}}^{\varepsilon-2\delta}(\cdot \| \cdot) - \log \frac{1-\varepsilon}{\delta^2}$
- → Moderate deviation regime : $\varepsilon \to 0, R = I(X:B) - a_n$ [Chubb *et al.*'17]

Implications

One Shot | I.I.D. Asymptotics

One Shot

$$\varepsilon \le \mathrm{Tr}[\rho_{XB} \wedge (M-1)\rho_X \otimes \rho_B]$$

$$\varepsilon \le \mathrm{e}^{-E^{\downarrow}(\log M)}$$

$$M \ge D_{\mathrm{h}}^{\varepsilon-\delta}(\rho_{XB} || \rho_X \otimes \rho_B) + \log \delta$$

The Tightest one-shot capacity so far

I.I.D. Asymptotics

Large Deviation Regime

$$\varepsilon \le \mathrm{e}^{-nE^{\downarrow}(R)}, \forall n \in \mathbb{N}$$

Moderate Deviation Regime

$$\varepsilon \le \mathrm{e}^{-\frac{na_n^2}{V(X:B)}[1-o(a_n)]}$$

$(R = I(X:B) - a_n$, for $a_n \to 0, na_n^2 \to \infty)$

Moderate Deviation Regime

$$M \ge nI(X:B) + \sqrt{2V(X:B)}a_n - o(a_n)$$

$(\varepsilon = \exp\{na_n^2\}$, for $a_n \to 0, na_n^2 \to \infty)$

Small Deviation Regime

$$M \ge nI(X:B) + \sqrt{nV(X:B)}\Phi^{-1}(\varepsilon) - \frac{1}{2}\log n$$

On Hypothesis-Testing Divergence

- A Conjecture:

Second-order asymptotics for quantum hypothesis testing

Ke Li

Ann. Statist. 42(1): 171-189 (February 2014). DOI: 10.1214/13-AOS1185

$$D_{\mathrm{h}}^{\varepsilon}(\rho^{\otimes n}||\sigma^{\otimes n}) \geq nD(\rho||\sigma) + \sqrt{nV(X:B)}\Phi^{-1}(\varepsilon) + \frac{1}{2}\log n - O(1)$$

- If so, then the *best possible* achievable third-order coding rate:

(for general channels)

$$M \geq nI(X:B) + \sqrt{nV(X:B)}\Phi^{-1}(\varepsilon) - O(1)$$

Such a rate is already optimal for some classical channels!

A General Recipe for One-Shot QIT

Quantum Information-Theoretic Tasks

↓

Reduction to a one-shot
quantum hypothesis testing (QHT) problem

↓

One-shot characterizations of QHT
and its asymptotic expansions ← Bottleneck!

With Entanglement Assistance

Proof of Entanglement Assistance (1/2)

$A_1 \dots A_M$ with Alice
$R_1 \dots R_M$ with Bob

- Preparation: $\theta_{R_1A_1} \otimes \cdots \otimes \theta_{R_mA_m} \otimes \cdots \otimes \theta_{R_MA_M}$
- Encoding: for each $m \in [M]$, send system A_m.
- The channel output states for each $m \in [M]$ is:

$$\rho^m_{R^M B} = \theta_R^{(m-1)} \otimes \mathcal{N}_{A\to B}(\theta_{R_mA_m}) \otimes \theta_R^{(M-m)}, \qquad m \in [1:M]$$

Position-based encoding
[Anshu–Jain–Warsi'19]

- If tracing out $R_1 \cdots R_M$ but NOT R_m,

$$\mathrm{Tr}_{[M]\backslash\{m\}}[\rho^m_{R^M B}] = \mathcal{N}_{A\to B}(\theta_{R_mA_m})$$

$$\mathrm{Tr}_{[M]\backslash\{m\}}[\rho^{\bar{m}}_{R^M B}] = \theta_{R_m} \otimes \mathcal{N}_{A\to B}(\theta_{A_m}), \qquad \forall \bar{m} \neq m$$

Proof of Entanglement Assistance (2/2)

- PGM: $$\Pi^m_{RM_B} := \frac{\rho^m_{RM_B}}{\rho^m_{RM_B} + \sum_{\bar{m}\neq m} \rho^{\bar{m}}_{RM_B}}, \qquad m \in [1:M]$$

1. For transmitting each m, relate its error to '$\wedge$':

$$\mathrm{Tr}\left[\rho^m_{RM_B}\left(I - \Pi^m_{RM_B}\right)\right] = \mathrm{Tr}\left[\rho^m_{RM_B} \frac{\sum_{\bar{m}\neq m} \rho^{\bar{m}}_{RM_B}}{\rho^m_{RM_B} + \sum_{\bar{m}\neq m} \rho^{\bar{m}}_{RM_B}}\right] \leq \mathrm{Tr}\left[\rho^m_{RM_B} \wedge \sum_{\bar{m}\neq m} \rho^{\bar{m}}_{RM_B}\right]$$

2. Conditional $\mathbb{E}$, tracing out $R_1 \cdots R_M$ but NOT R_m

$$\Rightarrow \mathrm{Tr}\left[\mathcal{N}_{A\to B}(\theta_{RA}) \wedge (M-1)\theta_R \otimes \mathcal{N}_{A\to B}(\theta_A)\right]$$

□

One-Shot Quantum Packing Lemma

Theorem. Let ρ_{AB} and τ_A be density operators. Define, for $m \in [M]$,

$$\omega^m_{A_1 \dots A_M B} \coloneqq \tau_{A_1} \otimes \cdots \otimes \rho_{A_m B} \otimes \cdots \otimes \tau_{A_M}.$$

Then, there exists a measurement:

$$\Pi^m_{A_1 \dots A_M B} = \frac{\omega^m_{A_1 \dots A_M B}}{\sum_{\bar{m} \neq m} \omega^{\bar{m}}_{A_1 \dots A_M B}}, m \in [M]$$

satisfying for each $m \in [M]$,

$$\mathrm{Tr}\left[\omega^m_{A_1 \dots A_M B}\left(I - \Pi^m_{A_1 \dots A_M B}\right)\right] \leq \mathrm{Tr}[\rho_{AB} \wedge (M-1)\tau_A \otimes \rho_B].$$

Applications (1/2)

Information-theoretic tasks	**One-shot achievability**	Bounds on coding error / Bounds on coding size
Point-to-point quantum channel	$\varepsilon \leq \mathrm{Tr}\left[\mathcal{N}_{\mathsf{A}\to\mathsf{B}}(\theta_{\mathsf{XA}}) \wedge (M-1)\theta_{\mathsf{X}} \otimes \mathcal{N}_{\mathsf{A}\to\mathsf{B}}(\theta_{\mathsf{A}})\right]$	$\varepsilon \leq \mathrm{e}^{-\sup_{\alpha\in(1/2,1)} \frac{1-\alpha}{\alpha}\left(I^{\downarrow}_{2-1/\alpha}(\mathsf{X}:\mathsf{B})_{\mathcal{N}_{\mathsf{A}\to\mathsf{B}}(\theta_{\mathsf{XA}})} - \log M\right)}$ $\log M \geq I_{\mathrm{h}}^{\varepsilon-\delta}(\mathsf{X}:\mathsf{B})_{\mathcal{N}_{\mathsf{A}\to\mathsf{B}}(\theta_{\mathsf{XA}})} - \log\frac{1}{\delta}$
Entanglement-assisted point-to-point quantum channel	$\varepsilon \leq \mathrm{Tr}\left[\mathcal{N}_{\mathsf{A}\to\mathsf{B}}(\theta_{\mathsf{RA}}) \wedge (M-1)\theta_{\mathsf{R}} \otimes \mathcal{N}_{\mathsf{A}\to\mathsf{B}}(\theta_{\mathsf{A}})\right]$	$\varepsilon \leq \mathrm{e}^{-\sup_{\alpha\in(1/2,1)} \frac{1-\alpha}{\alpha}\left(I^{\downarrow}_{2-1/\alpha}(\mathsf{R}:\mathsf{B})_{\mathcal{N}_{\mathsf{A}\to\mathsf{B}}(\theta_{\mathsf{RA}})} - \log M\right)}$ $\log M \geq I_{\mathrm{h}}^{\varepsilon-\delta}(\mathsf{R}:\mathsf{B})_{\mathcal{N}_{\mathsf{A}\to\mathsf{B}}((\theta_{\mathsf{RA}})} - \log\frac{1}{\delta}$
Quantum channel with casual state information	$\varepsilon \leq \mathrm{Tr}\left[\mathcal{N}_{\mathsf{AS}\to\mathsf{B}}(\theta_{\mathsf{UAS}}) \wedge (M-1)\theta_{\mathsf{U}} \otimes \mathcal{N}_{\mathsf{AS}\to\mathsf{B}}(\theta_{\mathsf{AS}})\right]$ $\forall\, \theta_{\mathsf{UAS}} : \mathrm{Tr}_{\mathsf{A}}[\theta_{\mathsf{UAS}}] = \theta_{\mathsf{U}} \otimes \vartheta_{\mathsf{S}}$	$\varepsilon \leq \mathrm{e}^{-\sup_{\alpha\in(1/2,1)} \frac{1-\alpha}{\alpha}\left(I^{\downarrow}_{2-1/\alpha}(\mathsf{U}:\mathsf{B})_{\mathcal{N}_{\mathsf{AS}\to\mathsf{B}}((\theta_{\mathsf{UAS}})} - \log M\right)}$ $\log M \geq I_{\mathrm{h}}^{\varepsilon-\delta}(\mathsf{U}:\mathsf{B})_{\mathcal{N}_{\mathsf{AS}\to\mathsf{B}}(\theta_{\mathsf{UAS}})} - \log\frac{1}{\delta}$
Entanglement-assisted quantum channel with casual state information	$\varepsilon \leq \mathrm{Tr}\left[\mathcal{N}_{\mathsf{AS}\to\mathsf{B}}(\theta_{\mathsf{RAS}}) \wedge (M-1)\theta_{\mathsf{R}} \otimes \mathcal{N}_{\mathsf{AS}\to\mathsf{B}}(\theta_{\mathsf{AS}})\right]$ $\forall\, \theta_{\mathsf{RAS}} : \mathrm{Tr}_{\mathsf{A}}[\theta_{\mathsf{RAS}}] = \theta_{\mathsf{R}} \otimes \vartheta_{\mathsf{S}}$	$\varepsilon \leq \mathrm{e}^{-\sup_{\alpha\in(1/2,1)} \frac{1-\alpha}{\alpha}\left(I^{\downarrow}_{2-1/\alpha}(\mathsf{R}:\mathsf{B})_{\mathcal{N}_{\mathsf{AS}\to\mathsf{B}}((\theta_{\mathsf{RAS}})} - \log M\right)}$ $\log M \geq I_{\mathrm{h}}^{\varepsilon-\delta}(\mathsf{R}:\mathsf{B})_{\mathcal{N}_{\mathsf{AS}\to\mathsf{B}}(\theta_{\mathsf{RAS}})} - \log\frac{1}{\delta}$

Applications (2/2)

Broadcast quantum channel	$\varepsilon_{\mathsf{B}} \leq \operatorname{Tr}\left[\operatorname{Tr}_{\mathsf{C}}\left[\mathcal{N}_{\mathsf{A}\to\mathsf{BC}}(\theta_{\mathsf{UA}})\right] \wedge (M_{\mathsf{B}}-1)\theta_{\mathsf{U}} \otimes \operatorname{Tr}_{\mathsf{C}}\left[\mathcal{N}_{\mathsf{A}\to\mathsf{BC}}(\theta_{\mathsf{A}})\right]\right]$ $\varepsilon_{\mathsf{C}} \leq \operatorname{Tr}\left[\operatorname{Tr}_{\mathsf{B}}\left[\mathcal{N}_{\mathsf{A}\to\mathsf{BC}}(\theta_{\mathsf{VA}})\right] \wedge (M_{\mathsf{C}}-1)\theta_{\mathsf{V}} \otimes \operatorname{Tr}_{\mathsf{B}}\left[\mathcal{N}_{\mathsf{A}\to\mathsf{BC}}(\theta_{\mathsf{A}})\right]\right] \ \forall\, \theta_{\mathsf{UVA}} : \operatorname{Tr}_{\mathsf{A}}\left[\theta_{\mathsf{UVA}}\right] = \theta_{\mathsf{U}} \otimes \theta_{\mathsf{V}}$	
Entanglement-assisted broadcast quantum channel	$\varepsilon_{\mathsf{B}} \leq \operatorname{Tr}\left[\operatorname{Tr}_{\mathsf{C}}\left[\mathcal{N}_{\mathsf{A}\to\mathsf{BC}}\left(\theta_{\mathsf{R_B A}}\right)\right] \wedge (M_{\mathsf{B}}-1)\theta_{\mathsf{R_B}} \otimes \operatorname{Tr}_{\mathsf{C}}\left[\mathcal{N}_{\mathsf{A}\to\mathsf{BC}}(\theta_{\mathsf{A}})\right]\right]$ $\varepsilon_{\mathsf{C}} \leq \operatorname{Tr}\left[\operatorname{Tr}_{\mathsf{B}}\left[\mathcal{N}_{\mathsf{A}\to\mathsf{BC}}\left(\theta_{\mathsf{R_C A}}\right)\right] \wedge (M_{\mathsf{C}}-1)\theta_{\mathsf{R_C}} \otimes \operatorname{Tr}_{\mathsf{B}}\left[\mathcal{N}_{\mathsf{A}\to\mathsf{BC}}(\theta_{\mathsf{A}})\right]\right] \ \forall \theta_{\mathsf{R_B R_C A}} : \operatorname{Tr}_{\mathsf{A}}\left[\theta_{\mathsf{R_B R_C A}}\right] = \theta_{\mathsf{R_B}} \otimes \theta_{\mathsf{R_C}}$	
Multiple-access quantum channel	$\varepsilon \leq \operatorname{Tr}\left[\rho_{\mathsf{XYC}} \bigwedge \left((M_{\mathsf{A}}-1)\rho_{\mathsf{X}} \otimes \rho_{\mathsf{YC}} + (M_{\mathsf{B}}-1)\rho_{\mathsf{Y}} \otimes \rho_{\mathsf{XC}} + (M_{\mathsf{A}}-1)(M_{\mathsf{B}}-1)\rho_{\mathsf{X}} \otimes \rho_{\mathsf{Y}} \otimes \rho_{\mathsf{C}}\right)\right]$ $\rho_{\mathsf{XYC}} := \mathcal{N}_{\mathsf{AB}\to\mathsf{C}}\left(\theta_{\mathsf{XA}} \otimes \theta_{\mathsf{YB}}\right)$	
Entanglement-assisted multiple-access quantum channel	$\varepsilon \leq \operatorname{Tr}\left[\rho_{\mathsf{R_A R_B C}} \bigwedge \left((M_{\mathsf{A}}-1)\rho_{\mathsf{R_A}} \otimes \rho_{\mathsf{R_B C}} + (M_{\mathsf{B}}-1)\rho_{\mathsf{R_B}} \otimes \rho_{\mathsf{R_A C}} + (M_{\mathsf{A}}-1)(M_{\mathsf{B}}-1)\rho_{\mathsf{R_A}} \otimes \rho_{\mathsf{R_B}} \otimes \rho_{\mathsf{C}}\right)\right]$ $\rho_{\mathsf{R_A R_B C}} := \mathcal{N}_{\mathsf{AB}\to\mathsf{C}}\left(\theta_{\mathsf{R_A A}} \otimes \theta_{\mathsf{R_B B}}\right)$	
Classical data compression with quantum side information	$\varepsilon \leq \operatorname{Tr}\left[\rho_{\mathsf{XB}} \bigwedge \left(\frac{1}{M}\mathbb{1}_{\mathsf{X}} \otimes \rho_{\mathsf{B}}\right)\right]$	$\varepsilon \leq \mathrm{e}^{-\sup_{\alpha\in(1/2,1)} \frac{1-\alpha}{\alpha}\left(\log M - H^{\downarrow}_{2-1/\alpha}(\mathsf{X}\,\vert\,\mathsf{B})_{\rho}\right)}$ $\log M \leq H_{\mathrm{h}}^{\varepsilon-\delta}(\mathsf{X}\,\vert\,\mathsf{B})_{\rho} + \log\frac{1}{\delta}$

A Strengthened Bound

$$\mathrm{Tr}\left[A\frac{B}{A+B}\right] \le \frac{\mathrm{Tr}[A \vee B]}{\mathrm{Tr}[A+B]}\,\mathrm{Tr}[A \wedge B] \le \mathrm{Tr}[A \wedge B]$$

- $A \vee B := \frac{1}{2}(A + B + |A - B|)$; $A \wedge B := \frac{1}{2}(A + B - |A - B|)$
- An idea of the data processing inequality [Sason–Verdu'18], [Renes'17]

$$\varepsilon \le \left(1 - \frac{1}{M}\mathrm{Tr}[\rho_{XB} \wedge (M-1)\rho_X \otimes \rho_B]\right)\mathrm{Tr}[\rho_{XB} \wedge (M-1)\rho_X \otimes \rho_B]$$

- Does it provide a simple proof for the upper bound of strong converse exponent [Mosonyi–Ogawa'17]?

Conclusions

- A conceptually simpler and intuitive one-shot achievability proof via the pretty-good measurement
- Clean and tighter one-shot expressions
- Sharpening many existing results
- → Applications of converses for covering-type problems
- Bounding M: tighter achievable one-shot capacities
- Bounding ε: not achieving the tightest error exponent yet

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A Strengthened Bound (1/2)

- The bound should be not bad for $\log M < I(X{:}B)$.
- For $\log M \geq I_{\max}(X{:}B)$, then $\varepsilon \leq \mathrm{Tr}[\rho_{XB} \wedge M\rho_X \otimes \rho_B] = 1$.
- [Wilde *et al.*'14], [Mosonyi–Ogawa'17]: $\varepsilon \geq 1 - \mathrm{e}^{I_{\max}(X:B)}/M$
- A Trace inequality for noncommutative parallel sum:

$$\mathrm{Tr}\left[A\frac{B}{A+B}\right] \leq \frac{\mathrm{Tr}[A \vee B]}{\mathrm{Tr}[A+B]}\,\mathrm{Tr}[A \wedge B] \leq \mathrm{Tr}[A \wedge B]$$

  - $A \vee B \coloneqq \frac{1}{2}(A + B + |A - B|)$; $A \wedge B \coloneqq \frac{1}{2}(A + B - |A - B|)$
  - An idea of the data processing inequality [Sason–Verdu'18], [Renes'17]

A Strengthened Bound (2/2)

- A tighter one-shot bound (tighter for large M):

$$\varepsilon \leq \left(1 - \frac{1}{M}\mathrm{Tr}[\rho_{XB} \wedge (M-1)\rho_X \otimes \rho_B]\right)\mathrm{Tr}[\rho_{XB} \wedge (M-1)\rho_X \otimes \rho_B]$$

- For $\log M \geq I_{\max}(X\colon B)$, then $\varepsilon \leq 1 - 1/M$.
- Does it provide a simple proof for the upper bound of strong converse exponent [Mosonyi–Ogawa'17]?