Resource Theory of Superposition

Superposition States Transformation & Golden States

Gökhan Torun

joint works with Hüseyin T. Şenyaşa, and Ali Yildiz

- G.Torun, H.T.Şenyaşa, and A.Yildiz, PRA 103, 032416 (2021)
- H.T.Şenyaşa and G.Torun, PRA 105, 042410 (2022)

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Ingredients	Coherence	Superposition
Basis States	$\{\|k\rangle : k = 0, 1, \ldots, d-1\}$ reference basis; complete and orthonormal	$\{\|c_i\rangle : i = 0, 1, \ldots, d-1\}$ normalized and linear independent; not necessarily orthogonal
Free States ($\mathcal{F}$)	$\rho = \sum_k p_k \|k\rangle\langle k\|$ $p_k \in [0,1]$ such that $\sum_k p_k = 1$	$\varrho = \sum_i \rho_i \|c_i\rangle\langle c_i\|$ $\rho_i \geq 0$, probability distribution
Resource States ($\mathcal{R}$)	any state $\sigma \notin \mathcal{F}$	any state $\varsigma \notin \mathcal{F}$
Free Operations ($\mathcal{O}$)	incoherent operation (IO); MIO, DIO, IO, SIO, PIO	superposition-free $\left(K_n \varrho K_n^\dagger \in \mathcal{F} \text{ for all } \varrho \in \mathcal{F}\right)$

◇ T.Baumgratz *et al.* PRL, 113, 140401 (2014) ◇ T.Theurer *et al.* PRL, 119, 230401 (2017)

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$$|\psi\rangle = \sum_{i=0}^{d-1} \psi_i |c_i\rangle \, ; \quad \left[\psi_i \in \mathbb{C},\ G_{ij} = \langle c_i | c_j\rangle \in \mathbb{C},\ \langle\psi|\psi\rangle = \sum_{i,j=0}^{d-1} \psi_i^* G_{ij} \psi_j = 1\right]$$

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The Problem #1

$$|\psi\rangle = \sum_{i=1}^{d} \psi_i |c_i\rangle \xrightarrow{\mathcal{O}} |\varphi\rangle = \sum_{i=1}^{d} \varphi_i |c_i\rangle \quad ?$$

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Coherence transformation: $\sum_{k=1}^{d} \psi_k |k\rangle \xrightarrow{\mathcal{O}} \sum_{k=1}^{d} \phi_k |k\rangle$

Majorization: $\boxed{\sum_{k=1}^{r} |\psi_k|^2 \leq \sum_{k=1}^{r} |\phi_k|^2}$ for any $r \in [1, d]$

- $K_n = \sqrt{p_n} \sum_{k=1}^{d} \left(\frac{\varphi_{f_n(k)}}{\psi_k}\right) |f_n(k)\rangle\langle k|$; $\sum_{n=1}^{d} K_n^\dagger K_n = \mathbb{1}$
- $\sum_{n=1}^{d} p_n \varphi_{f_n(k)} = \psi_k \;\; (k = 1, 2, \ldots, d)$

◇ S.Du *et al.* PRA 91, 052120 (2015) ◇ G.Torun *et al.* PRA 97, 052331 (2018)

The Problem #1

$$|\psi\rangle = \sum_{i=1}^{d} \psi_i |c_i\rangle \xrightarrow{\mathcal{O}} |\varphi\rangle = \sum_{i=1}^{d} \varphi_i |c_i\rangle \quad ?$$

Setting the approach:

$$K_n = \sum_{k=1}^{d} c_{k,n} \frac{|c_{f_n(k)}\rangle\langle c_k^{\perp}|}{\langle c_k^{\perp}|c_k\rangle}; \left[c_{k,n} = \sqrt{p_n}\left(\frac{\varphi_{f_n(k)}}{\psi_k}\right), \langle c_i^{\perp}|c_j\rangle = \zeta_i \delta_{ij} \in \mathbb{C} \right]$$

The superposition-free operator K_n gives $|\varphi\rangle$ with the probability p_n:

$$\boxed{K_n|\psi\rangle} = \sqrt{p_n} \sum_{k=1}^{d} \varphi_{f_n(k)} |c_{f_n(k)}\rangle = \boxed{\sqrt{p_n}|\varphi\rangle} \quad \left(p_n \geq 0 \ \& \ \sum_n p_n = 1\right).$$

completeness: $\boxed{\sum_{n=1}^{d} K_n^{\dagger} K_n + \sum_{m=d+1}^{2d} F_m^{\dagger} F_m = \mathbb{1}} \quad \left(F_m|\psi\rangle = 0, \boxed{\sum_{n=1}^{d} K_n^{\dagger} K_n \leq \mathbb{1}} \right).$

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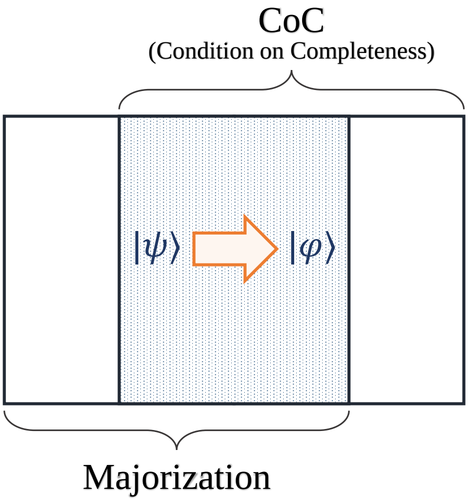
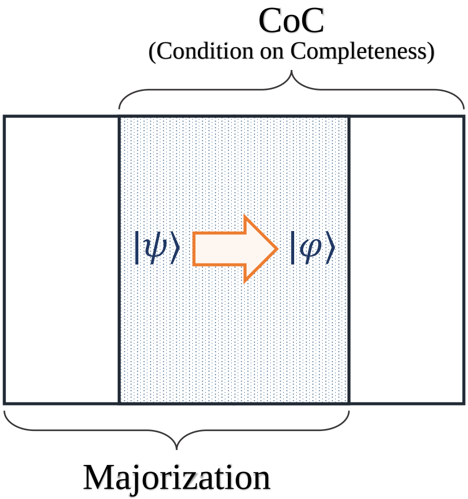
The quest for:

- ☐ Index functions $\{f_n(k)\}_{k=1}^{d}$
- ☐ Probabilities $\{p_n\}$
- ☐ Superposition-free Kraus operators $\{K_n\}$
- ■ Conditions for (deterministic) superposition-free transformations



◇ G.Torun *et al.* PRA, 103, 032416 (2021).



Majorization: $\boxed{\sum_{i=1}^{r} \tilde{\psi}_i \leq \sum_{i=1}^{r} \tilde{\varphi}_i}$ for any $r \in [1, d]$

$$\tilde{z}_i := \sum_{j=1}^{d} z_i^* z_j \langle c_i | c_j \rangle, \; i = 1, 2, \ldots, d \; (z \equiv \psi, \varphi)$$

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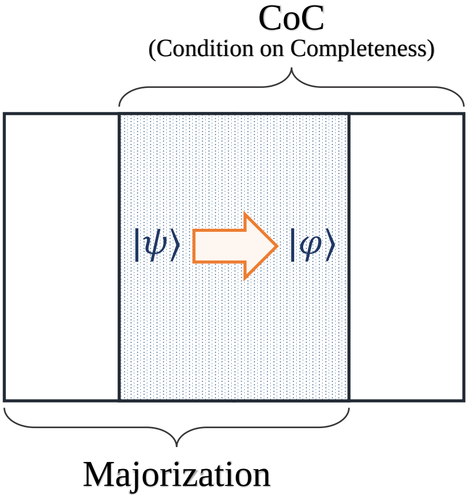

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CoC: $\boxed{\sum_{i=1}^{d} p_i \omega_{ij} \leq \psi_j^2}$ for $j = 2,3,\ldots,d$

$$\begin{pmatrix} \omega_{i1} \\ \vdots \\ \omega_{id} \end{pmatrix} = P_i \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_d \end{pmatrix} \text{ and } \sum_{n=1}^{d} p_n \tilde{\varphi}_{f_n(k)} = \tilde{\psi}_k \quad (k = 1,2,\ldots,d)$$

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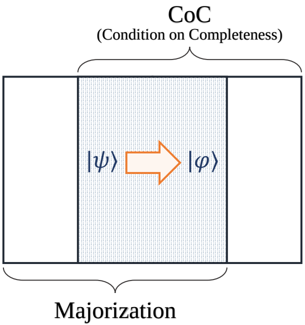
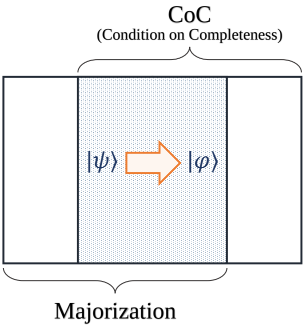


$\|\psi\rangle \xrightarrow{\mathcal{O}} \|\varphi\rangle$ with $\langle c_1\|c_2\rangle = 1/2$	Maj.	CoC
$\frac{1}{\sqrt{13}}(3\|c_1\rangle + \|c_2\rangle) \xrightarrow{\mathcal{O}} \frac{1}{\sqrt{13}}(4\|c_1\rangle - \|c_2\rangle)$	✓	✗
$\frac{1}{\sqrt{21}}(4\|c_1\rangle + \|c_2\rangle) \xrightarrow{\mathcal{O}} \frac{1}{\sqrt{7}}(3\|c_1\rangle - \|c_2\rangle)$	✗	✓
$\frac{1}{\sqrt{7}}(3\|c_1\rangle - \|c_2\rangle) \xrightarrow{\mathcal{O}} \frac{1}{\sqrt{13}}(4\|c_1\rangle - \|c_2\rangle)$	✓	✓



$\vert\psi\rangle \xrightarrow{\mathcal{O}} \vert\varphi\rangle$ with $\langle c_1\vert c_2\rangle = 1/2$	Maj.	CoC
$\frac{1}{\sqrt{13}}(3\vert c_1\rangle + \vert c_2\rangle) \xrightarrow{\mathcal{O}} \frac{1}{\sqrt{13}}(4\vert c_1\rangle - \vert c_2\rangle)$	✓	✗
$\frac{1}{\sqrt{21}}(4\vert c_1\rangle + \vert c_2\rangle) \xrightarrow{\mathcal{O}} \frac{1}{\sqrt{7}}(3\vert c_1\rangle - \vert c_2\rangle)$	✗	✓
$\frac{1}{\sqrt{7}}(3\vert c_1\rangle - \vert c_2\rangle) \xrightarrow{\mathcal{O}} \frac{1}{\sqrt{13}}(4\vert c_1\rangle - \vert c_2\rangle)$	✓	✓

$$\left\{K_1 = \sqrt{\tfrac{7p_1}{13}}\left(\tfrac{4\alpha_1}{3}|c_1\rangle\langle c_1^\perp| + \alpha_2|c_2\rangle\langle c_2^\perp|\right); K_2 = -\sqrt{\tfrac{7p_2}{13}}\left(\tfrac{\alpha_1}{3}|c_2\rangle\langle c_1^\perp| + 4\alpha_2|c_1\rangle\langle c_2^\perp|\right)\right\} \quad \boxed{p_1 = \frac{209}{210},\ p_2 = \frac{1}{210}}$$

$$\left\{F_3 = -\sqrt{\tfrac{11}{26}}\left(\tfrac{\alpha_1}{3}|c_1\rangle\langle c_1^\perp| + \alpha_2|c_1\rangle\langle c_2^\perp|\right); F_4 = 0\right\} \quad \boxed{\sum_{i=1}^{2} K_i^\dagger K_i + F_3^\dagger F_3 = \mathbb{1}} \quad \boxed{\sum_{i=1}^{2} K_i^\dagger K_i \leq \mathbb{1}} \quad \left(\alpha_i \equiv 1/\langle c_i^\perp|c_i\rangle\right).$$

Majorization: $\boxed{\sum_{i=1}^{r} \tilde{\psi}_i \leq \sum_{i=1}^{r} \tilde{\varphi}_i}$ for any $r \in [1, d]$

$$\tilde{z}_i := \sum_{j=1}^{d} z_i^* z_j \langle c_i | c_j \rangle, \; i = 1, 2, \ldots, d \; (z \equiv \psi, \varphi)$$

CoC: $\boxed{\sum_{i=1}^{d} p_i \omega_{ij} \leq \psi_j^2}$ for $j = 2, 3, \ldots, d$

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Orthonormal limit: $\langle c_i | c_j \rangle = 0,\ \{|c_i\rangle\} \to \{|j\rangle\}_{j=1}^{d}$.

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- $\left\{K_n = \sqrt{p_n} \sum_{k=1}^{d} \left(\frac{\varphi_{f_n(k)}}{\psi_k}\right) |f_n(k)\rangle\langle k|,\ F_m = 0\right\}$
- $\sum_{n=1}^{d} K_n^\dagger K_n = \mathbb{1}\ ;\ \sum_{n=1}^{d} p_n \varphi_{f_n(k)} = \psi_k\ (k = 1, \ldots, d)$

Permutations $\{P_i\}$ ... Consider $d = 3$ ... Already $|\psi_1|^2 \leq |\varphi_1|^2$ and $|\psi_3|^2 \geq |\varphi_3|^2$ ... However, it is either $|\psi_2|^2 \leq |\varphi_2|^2$ or $|\psi_2|^2 \geq |\varphi_2|^2$.

◇ If $\psi_2 \leq \varphi_2$: $\{P_i\} = \{\mathbb{1}, |1\rangle \leftrightarrow |3\rangle, |2\rangle \leftrightarrow |3\rangle\}$.

◇ If $\psi_2 \geq \varphi_2$: $\{P_i\} = \{\mathbb{1}, |1\rangle \leftrightarrow |3\rangle, |1\rangle \leftrightarrow |2\rangle\}$.

◇ PRA 97, 052331 (2018)

The Problem #2

$$\text{Coherence}: |\Phi_d\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}|i\rangle \; ; \; \text{Superposition}: ?$$

◇ H.T.Şenyaşa and G.Torun, PRA 105, 042410 (2022).

The Problem #2

$$\text{Coherence}: |\Phi_d\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}|i\rangle \; ; \; \text{Superposition}: ?$$

Let G be a Gram matrix that represents the inner product settings of superposition constructed by a set of linearly independent basis states $\{|c_i\rangle\}_{i=1}^{d}$, that is, $G_{ij} = \langle c_i|c_j\rangle$. Then a maximal state $\vec{\psi}$ is necessarily an eigenvector of G corresponding to the minimum eigenvalue $\lambda_{\min}$ and the components of $\vec{\psi}$ must satisfy $\tilde{\psi}_i = 1/d$ for all $i = 1, 2, \ldots, d$ — Proposition 1.

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$$\left(\tilde{\psi}_i := \textstyle\sum_{j=1}^{d} \psi_i^* \psi_j G_{ij} \;\text{ for }\; |\psi\rangle = \sum_{i=1}^{d} \psi_i |c_i\rangle\right)$$

Corollary. The form of the (candidate) maximal state dictated by *Proposition 1* is as follows

$$\vec{\psi} = \sqrt{\frac{1}{d\lambda_{\min}}}\left(e^{i\theta_1}, \ldots, e^{i\theta_d}\right)^{\mathsf{T}}.$$

$$|\psi\rangle = \sqrt{\frac{1}{d\lambda_{\min}}}\sum_{j=1}^{d} e^{i\theta_j}|c_j\rangle$$

◇ H.T.Şenyaşa and G.Torun, PRA 105, 042410 (2022).

Rayleigh quotient — also known as the Rayleigh-Ritz ratio — can be used to determine the upper and lower bounds of a quadratic equation under elliptic constraint. Let $\vec{v}$ and $\vec{u}$ be eigenvectors of a Gram matrix G corresponding to the minimum eigenvalue λ_{min} and maximum eigenvalue λ_{max}, respectively. Let $\vec{x}$ be an arbitrary vector under elliptic constraint represented by $\vec{x}^\dagger G\vec{x} = 1$. Then, for these three vectors $\{\vec{v}, \vec{u}, \vec{x}\}$ the Rayleigh quotient guarantees

$$R(G,\vec{v}) = \lambda_{\text{min}} \leq R(G,\vec{x}) \leq R(G,\vec{u}) = \lambda_{\text{max}}, \text{ where } \left[R(G,\vec{x}) = \frac{\vec{x}^\dagger G\vec{x}}{\vec{x}^\dagger \vec{x}}\right]$$

Given an initial state $\vec{\psi} = (e^{i\theta_1}\psi_1, \ldots, e^{i\theta_d}\psi_d)^\intercal$ and a final state $\vec{\phi} = (e^{i\gamma_1}\phi_1, \ldots, e^{i\gamma_d}\phi_d)^\intercal$, where $\theta_j, \gamma_j \in [0, 2\pi)$ and $\psi_j, \phi_j \in \mathbb{R}^{\geq 0}$, we have $\tilde{K}_n\vec{\psi} = \sqrt{p_n}\vec{\phi}$, where the probabilities are chosen to be $p_n = 1/d!$ for $n = 1, 2, \ldots, d!$.

$$\boxed{\sum_{i=1}^{d} \psi_i^2 \geq \sum_{i=1}^{d} \varphi_i^2}$$

For a maximal state, the left-hand side has to take the maximum value so that it always holds for an arbitrary final state. By invoking the Rayleigh quotient, we conclude that a maximal state is necessarily an eigenvector of the Gram matrix corresponding to the minimum eigenvalue.

Proposition 2. *Let G be a Gram matrix that represents the inner product settings of superposition constructed by a set of linearly independent basis states $\{|c_1\rangle, |c_2\rangle\}$. Then the eigenvector $\vec{\psi}$ corresponding to the minimum eigenvalue $\lambda_{\min}$ is a maximal state.*

. . . .

Consider the case $\langle c_1|c_2\rangle = s \in (-1,1)$. The Gram matrix has two eigenvalues $\lambda_1 = 1-s$ and $\lambda_2 = 1+s$, where the corresponding eigenvectors are $\vec{x}_1 = (1,-1)^{\intercal}$ and $\vec{x}_2 = (1,1)^{\intercal}$. Two different situations occur: $\lambda_{\min} = 1-s$ when $s \in [0,1)$ and $\lambda_{\min} = 1+s$ when $s \in (-1,0]$.

$$|\Psi_2^-\rangle = \frac{1}{\sqrt{2(1-s)}}\left(|c_1\rangle - |c_2\rangle\right) \text{ with } s \in [0,1) \left(\tilde{\psi}_1 = \tilde{\psi}_2 = \frac{1}{2}\right)$$

$$|\Psi_2^+\rangle = \frac{1}{\sqrt{2(1+s)}}\left(|c_1\rangle + |c_2\rangle\right) \text{ with } s \in (-1,0]$$

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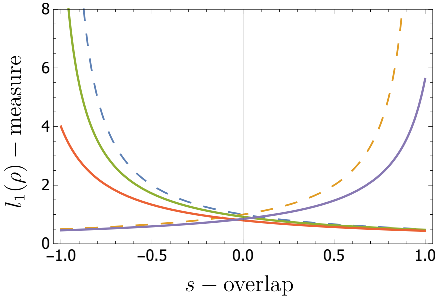

The dashed orange line, $l_1(\rho_{|\Psi_2^-\rangle}) = 1/(1-s)$; and the dashed blue line, $l_1(\rho_{|\Psi_2^+\rangle}) = 1/(1+s)$.

Setting $\psi_1 \equiv x \neq 0$ and $\psi_0 \equiv \eta x$ [for qubit systems we always have $\psi_0/\psi_1 = \eta \in \mathbb{R}\backslash\{0\}$], we then have $|\psi\rangle = x(\eta|c_0\rangle + |c_1\rangle)$. The solid green line, $l_1(\rho_{|\psi\rangle})$ for $\eta = 3/2$; the solid red line, $l_1(\rho_{|\psi\rangle})$ for $\eta = 2$; and the solid purple line, $l_1(\rho_{|\psi\rangle})$ for $\eta = -9/5$.

Inner product settings: $\{s_{12}, s_{13}, s_{23}\}$	$\{\lambda_{\min}, \vec{x}\}$	Range of s
$\{s, s, s\}$	$\{1+2s, (1,1,1)^{\mathsf{T}}\}$	$s \in (-\frac{1}{2}, 0]$
$\{-s, s, s\}$	$\{1-2s, (1,1,-1)^{\mathsf{T}}\}$	$s \in [0, \frac{1}{2})$
$\{s, -s, s\}$	$\{1-2s, (1,-1,1)^{\mathsf{T}}\}$	$s \in [0, \frac{1}{2})$
$\{s, s, -s\}$	$\{1-2s, (-1,1,1)^{\mathsf{T}}\}$	$s \in [0, \frac{1}{2})$
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$\{-s, -s, -s\}$	$\{1-2s, (1,1,1)^{\mathsf{T}}\}$	$s \in [0, \frac{1}{2})$
$\{s, is, -is\}$	$\{1-2s, (-i,i,1)^{\mathsf{T}}\}$	$s \in [0, \frac{1}{2})$

Consider $\langle c_1|c_2\rangle = -s$ and $\langle c_1|c_3\rangle = \langle c_2|c_3\rangle = s$ ...
Then, $\det(G) = 1 - 3s^2 - 2s^3 > 0$ implying $s \in (-1, \frac{1}{2})$ ...

$$\lambda_1 = \lambda_2 = 1 + s, \quad \lambda_3 = 1 - 2s\,;$$

$$\vec{x}_1 = (1,0,-1)^{\mathsf{T}},\ \vec{x}_2 = (1,-1,0)^{\mathsf{T}},\ \vec{x}_3 = (1,1,-1)^{\mathsf{T}}.$$

$$|\Psi_3^-\rangle = \frac{1}{\sqrt{3(1-2s)}}\Big(|c_1\rangle + |c_2\rangle - |c_3\rangle\Big),\ s \in [0, \frac{1}{2}).$$

Inner product settings: $\{s_{12}, s_{13}, s_{23}\}$	$\{\lambda_{\min}, \vec{x}\}$	Range of s
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$\{-s,-s,s\}$	$\{1+2s,(-1,1,1)^\intercal\}$	$s\in(-\frac{1}{2},0]$
$\{-s,-s,-s\}$	$\{1-2s,(1,1,1)^\intercal\}$	$s\in[0,\frac{1}{2})$
$\{s,is,-is\}$	$\{1-2s,(-i,i,1)^\intercal\}$	$s\in[0,\frac{1}{2})$

Consider $\langle c_1|c_2\rangle = -s$ and $\langle c_1|c_3\rangle = \langle c_2|c_3\rangle = s$ ...
Then, $\det(G) = 1-3s^2-2s^3 > 0$ implying $s\in(-1,\frac{1}{2})$ ...

$$\lambda_1=\lambda_2=1+s,\quad \lambda_3=1-2s\,;$$

$$\vec{x}_1=(1,0,-1)^\intercal,\ \vec{x}_2=(1,-1,0)^\intercal,\ \vec{x}_3=(1,1,-1)^\intercal.$$

$$|\Psi_3^-\rangle = \frac{1}{\sqrt{3(1-2s)}}\Big(|c_1\rangle+|c_2\rangle-|c_3\rangle\Big),\ s\in[0,\frac{1}{2}).$$

Proposition 3. The state

$$|\Psi_d^+\rangle := \frac{1}{\sqrt{d\,(1+(d-1)s)}}\sum_{j=1}^{d}|c_j\rangle,\quad s\in(\frac{1}{1-d},0],$$

is a maximal state for an equal and real inner product setting, that is, $\langle c_i|c_j\rangle = s$, $s\in\mathbb{R}$, for $i\neq j$.

G.Torun, arXiv:2209.03746

- $\{|c_i\rangle : i = 0, 1, \ldots, d-1\}$
- Gram matrix S with the elements $S_{ij} = \langle c_i | c_j \rangle$
- $\{|l_i\rangle : i = 0, 1, \ldots, d-1\}$ — Löwdin basis
- $\boldsymbol{C} := (c_0, c_1, \ldots, c_{d-1})^{\mathsf{T}}, \quad \boldsymbol{L} := (l_0, l_1, \ldots, l_{d-1})^{\mathsf{T}}$
- A general linear transformation T such that

  $\boldsymbol{L} = T\boldsymbol{C}, \quad \left(TT^{\dagger} = S^{-1}\right)$
- Which can be rewritten as

$$(\boldsymbol{L})_i = \sum_{j=0}^{d-1} \left(S^{-1/2}\right)_{ij} (\boldsymbol{C})_j \qquad \boxed{|l_i\rangle = \sum_{j=0}^{d-1} (S^{-1/2})_{ij} |c_j\rangle}$$

$$\left[S_{\text{diag}} = U^{\dagger} S U \;;\; S^{1/2} = U S_{\text{diag}}^{1/2} U^{\dagger} \;;\; S^{-1/2} = \left(S^{1/2}\right)^{-1}\right]$$

"Löwdin symmetric orthogonalization"

◇ P.Löwdin, J. Chem. Phys. 18, 365 (1950).

◇ J.G.Aiken, J.A.Erdos, and J.A.Goldstein, Int. J. Quantum Chem. 18, 1101 (1980).

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superposition state ⇄ coherent state

- $S_{ij} = \langle c_i|c_j\rangle = s \in (-1, 1)$
- Eigenvalues of S are found to be $\lambda_0 = \cdots = \lambda_{d-2} = 1 - s$ and $\lambda_{d-1} = 1 + (d-1)s$
- Setting

$$\mu := \frac{1}{d}\left[\frac{1}{\sqrt{1+(d-1)s}} + \frac{d-1}{\sqrt{1-s}}\right]$$

and

$$\kappa := \frac{1}{d}\left[\frac{1}{\sqrt{1+(d-1)s}} - \frac{1}{\sqrt{1-s}}\right]$$

- We get $|l_i\rangle = \mu|c_i\rangle + \kappa \sum_{\substack{j=0 \\ (j \neq i)}}^{d-1} |c_j\rangle \quad (i = 0, \ldots, d-1)$

The symmetric orthogonalization ensures $\sum_i \|c_i - l_i\|^2 = \min.$

superposition state ⇄ coherent state

$$|\psi\rangle = \sum_{i=0}^{d-1} \psi_i |c_i\rangle \xrightarrow{\text{LSO}^{\rightarrow}} |\bar{\psi}\rangle = \sum_{i=0}^{d-1} g_i(\psi, S)|l_i\rangle$$

$$|\bar{\psi}\rangle = \sum_{i=0}^{d-1} g_i(\psi, S)|l_i\rangle \xrightarrow{\text{LSO}^{\leftarrow}} |\psi\rangle = \sum_{i=0}^{d-1} \psi_i |c_i\rangle$$

$$g_i(\psi, S) = \sqrt{\lambda_{d-1}\lambda_0}\left[\Big(\mu + [d-2]\kappa\Big)\psi_i - \kappa \sum_{\substack{j=0 \\ (j\neq i)}}^{d-1} \psi_j\right]$$

$$\left(\psi_i \equiv \mu g_i(\psi, S) + \kappa \sum_{\substack{j=0 \\ (j\neq i)}}^{d-1} g_j(\psi, S) \;\; (i = 0, 1, \ldots, d-1)\right)$$

superposition state $\rightleftarrows$ coherent state

$$|\psi\rangle = \sum_{i=0}^{d-1} \psi_i |c_i\rangle \xrightarrow{\text{LSO}^{\rightarrow}} |\bar{\psi}\rangle = \sum_{i=0}^{d-1} g_i(\psi, S) |l_i\rangle$$

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maximally coherent state $\leftrightarrows$ state with maximal superposition

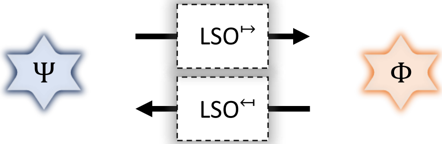
$$\frac{1}{\sqrt{2}}\left(|l_0\rangle + |l_1\rangle\right) \xrightarrow{\text{LSO}^{\leftarrow}} |\Psi_2^+\rangle = \frac{|c_0\rangle + |c_1\rangle}{\sqrt{2(1+s)}} \ ; \ s \in (-1, 0]$$

. . . .

$$\frac{1}{\sqrt{2}}\left(|l_0\rangle - |l_1\rangle\right) \xrightarrow{\text{LSO}^{\leftarrow}} |\Psi_2^-\rangle = \frac{|c_0\rangle - |c_1\rangle}{\sqrt{2(1-s)}} \ ; \ s \in [0, 1)$$

. . . .

$$|\Phi_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |l_i\rangle \xrightarrow{\text{LSO}^{\leftarrow}} |\Psi_d^+\rangle = \frac{|c_0\rangle + \cdots + |c_{d-1}\rangle}{\sqrt{d(1 + [d-1]s)}}$$



To conclude ...

- The basics of the resource theory of superposition ✓
- An explicit framework for the (deterministic) transformation of superposition states ✓
- States with maximal superposition, i.e., golden states for superposition ✓
- Results related with Löwdin symmetric orthogonalization promise the possibility of more ✓

- ☐ Mixed state transformations ?
- ☐ Superposition distillation ?
- ☐ Catalytic superposition transformations ?
- ☐ Effective role of Löwdin symmetric orthogonalization ?

Thanks for your time!