Overcoming entropic limitations on asymptotic state transformations through probabilistic protocols

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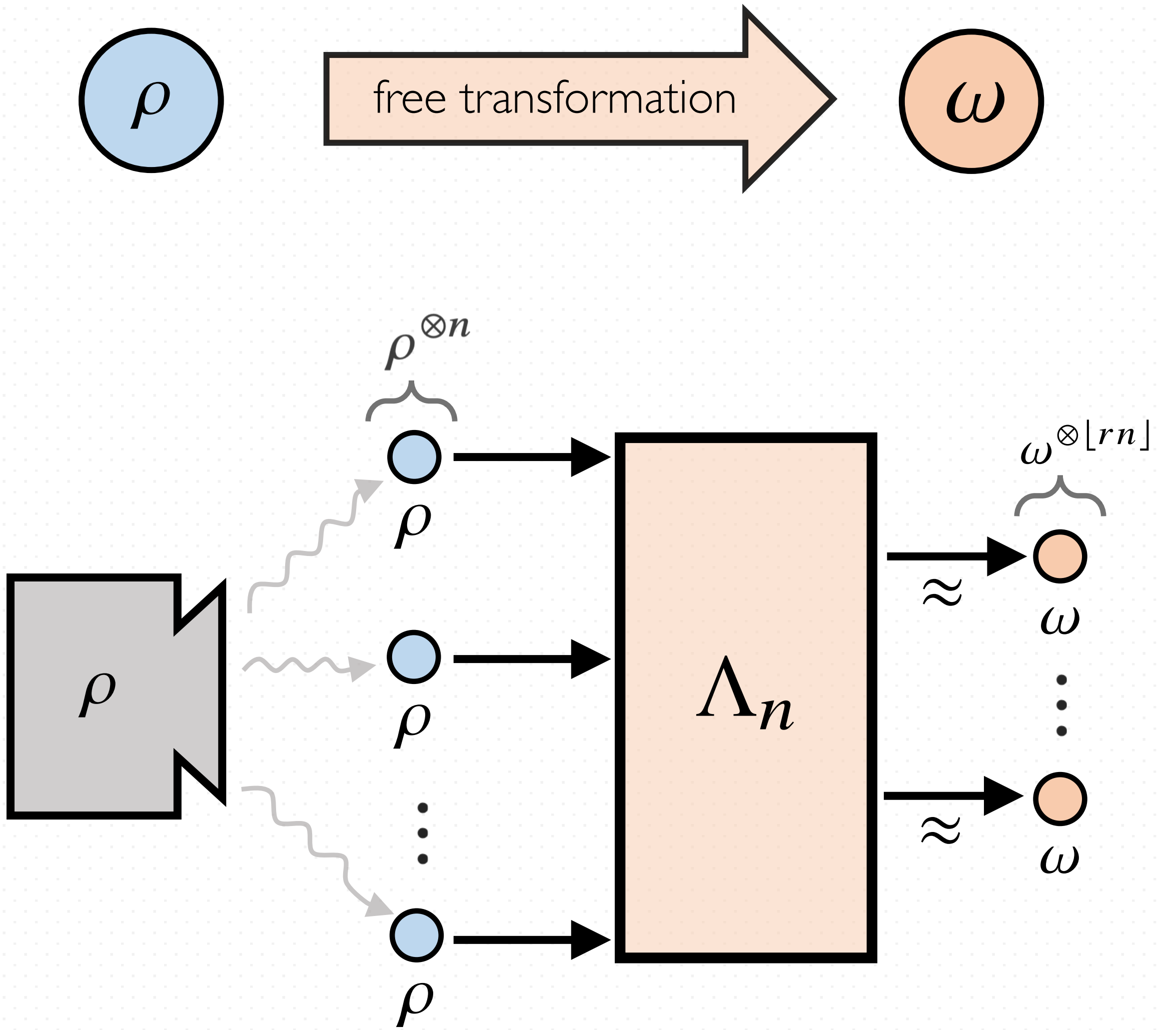
Manipulation of quantum states



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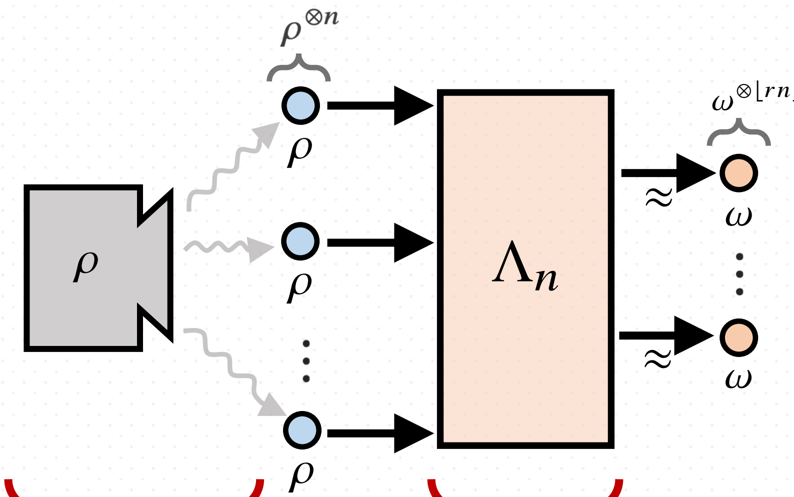
Asymptotic rates



cost in Shannon theory:
number of generated copies

cost in our framework:
size of quantum memory

asymptotic rate in Shannon theory

$$r(\rho \to \omega) := \sup \left\{ r \;\middle|\; \lim_{n\to\infty} \inf_{\Lambda_n \in \mathcal{O}} \left\| \Lambda_n(\rho^{\otimes n}) - \omega^{\otimes \lfloor rn \rfloor} \right\|_1 = 0 \right\},$$

	Practice	Shannon theory	**This work**
Number of copies	✓	✓	✗
Size of quantum memory	✓	✗	✓

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Asymptotic rates

asymptotic rate in Shannon theory

$$r(\rho \to \omega) := \sup \left\{ r \;\middle|\; \lim_{n\to\infty} \inf_{\Lambda_n \in \mathcal{O}\cap\mathrm{CPTP}} \left\| \Lambda_n(\rho^{\otimes n}) - \omega^{\otimes \lfloor rn \rfloor} \right\|_1 = 0 \right\}$$

asymptotic rate in our framework

$$r_{\mathrm{prob}}(\rho \to \omega) := \sup \left\{ r \;\middle|\; \lim_{n\to\infty} \inf_{\Lambda_n \in \mathcal{O}} \left\| \frac{\Lambda_n(\rho^{\otimes n})}{\mathrm{Tr}\,\Lambda_n(\rho^{\otimes n})} - \omega^{\otimes \lfloor rn \rfloor} \right\|_1 = 0 \right\}$$

strong converse rate

$$r^{\dagger}_{\mathrm{prob}}(\rho \to \omega) := \sup \left\{ r \;\middle|\; \liminf_{n\to\infty} \inf_{\Lambda_n \in \mathcal{O}} \frac{1}{2} \left\| \frac{\Lambda_n(\rho^{\otimes n})}{\mathrm{Tr}\,\Lambda_n(\rho^{\otimes n})} - \omega^{\otimes \lfloor rn \rfloor} \right\|_1 < 1 \right\}$$

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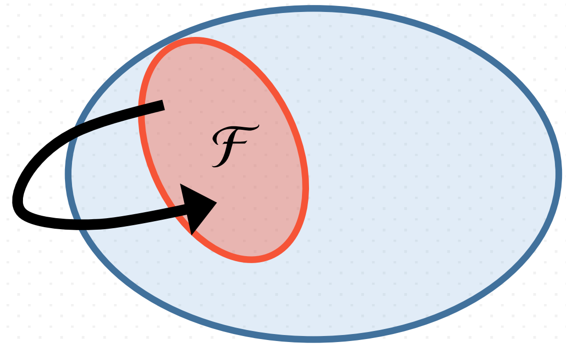
Divergences as resource monotones

quantum relative entropy

$$D(\rho\|\sigma) = \operatorname{Tr} \rho(\log \rho - \log \sigma)$$



free transformation

$$\sigma \in \mathcal{F} \;\; \Rightarrow \;\; \frac{\Lambda(\sigma)}{\operatorname{Tr} \Lambda(\sigma)} \in \mathcal{F}$$

relative entropy of a resource

$$D_{\mathcal{F}}(\rho) := \min_{\sigma \in \mathcal{F}} D(\rho\|\sigma)$$

$$D_{\mathcal{F}}(\Lambda(\rho)) \leq D_{\mathcal{F}}(\rho) \;\; \forall \Lambda \in \mathcal{O} \cap \mathrm{CPTP}$$

regularised relative entropy

$$D^{\infty}_{\mathcal{F}}(\rho) := \lim_{n \to \infty} \frac{1}{n} D_{\mathcal{F}}(\rho^{\otimes n})$$

$$D_{\mathcal{F}}\left(\frac{\Lambda(\rho)}{\operatorname{Tr} \Lambda(\rho)}\right) \not\leq D_{\mathcal{F}}(\rho) \;\; \forall \Lambda \in \mathcal{O}$$

$$r(\rho \to \omega) \leq \frac{D^{\infty}_{\mathcal{F}}(\rho)}{D^{\infty}_{\mathcal{F}}(\omega)}.$$

[Horodecki, QIC 1, 3 (2001)]

Max-relative entropy

max-relative entropy

[Datta, IEEE TIT 55, 2816 (2009)]

$$D_{\max}(\rho\|\sigma) := \log\inf\left\{\lambda\in\mathbb{R}_+ \;\middle|\; \rho\le\lambda\sigma\right\}$$
$$= \begin{cases}\log\left\|\sigma^{-1/2}\rho\sigma^{-1/2}\right\|_\infty & \operatorname{supp}(\rho)\subseteq\operatorname{supp}(\sigma)\\ \infty & \text{otherwise.}\end{cases}$$

$$D_{\max,\mathcal{F}}(\rho) := \min_{\sigma\in\mathcal{F}} D_{\max}(\rho\|\sigma)$$

smoothed regularised max-relative entropy

[Datta, IJQI 7, 475 (2009)]
[Brandão & Plenio, CMP 295, 829 (2010)]

$$D^{\infty,\bullet}_{\max,\mathcal{F}}(\rho) := \lim_{\varepsilon\to0}\lim_{n\to\infty}\min_{\frac12\|\rho'-\rho^{\otimes n}\|_1\le\varepsilon}\frac1n D_{\max,\mathcal{F}}(\rho') = D^{\infty}_{\mathcal{F}}(\rho),$$

$$D_{\mathcal{F}}(\rho) := \min_{\sigma\in\mathcal{F}} D(\rho\|\sigma)$$

Projective relative entropy

projective relative entropy (Hilbert projective metric)

$$\mathbb{D}_\Omega(\rho\|\sigma) := D_{\max}(\rho\|\sigma) + D_{\max}(\sigma\|\rho)$$

[Reeb et al., JMP 52, 082201 (2011)]

$$\mathbb{D}_{\Omega,\mathcal{F}}(\rho) := \min_{\sigma\in\mathcal{F}} \mathbb{D}_\Omega(\rho\|\sigma)$$

[Regula, PRL 128, 110505 (2022)]

$$\mathbb{D}_{\Omega,\mathcal{F}}\left(\frac{\Lambda(\rho)}{\operatorname{Tr}\Lambda(\rho)}\right) \leq \mathbb{D}_{\Omega,\mathcal{F}}(\rho) \;\; \forall \Lambda \in \mathcal{O}$$

$$\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho) := \lim_{n\to\infty} \frac{1}{n}\mathbb{D}_{\Omega,\mathcal{F}}\left(\rho^{\otimes n}\right)$$

$$\mathbb{D}^{\infty,\bullet}_{\Omega,\mathcal{F}}(\rho) := \lim_{\varepsilon\to 0} \lim_{n\to\infty} \min_{\frac{1}{2}\|\rho'-\rho^{\otimes n}\|_1\leq\varepsilon} \frac{1}{n}\mathbb{D}_{\Omega,\mathcal{F}}(\rho') = ?$$

Asymptotic equipartition property

Result 1

$$\mathbb{D}^{\infty,\bullet}_{\Omega,\mathcal{F}}(\rho) = D^{\infty}_{\mathcal{F}}(\rho)$$

$$\mathbb{D}^{\infty,\bullet}_{\Omega,\mathcal{F}}(\rho) := \lim_{\varepsilon\to 0} \lim_{n\to\infty} \min_{\frac{1}{2}\|\rho'-\rho^{\otimes n}\|_1 \le \varepsilon} \frac{1}{n} \mathbb{D}_{\Omega,\mathcal{F}}(\rho')$$

$$D^{\infty}_{\mathcal{F}}(\rho) := \lim_{n\to\infty} \frac{1}{n} D_{\mathcal{F}}(\rho^{\otimes n})$$

Main proof idea

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \min_{\frac{1}{2}\|\rho'-\rho^{\otimes n}\|_1 \le \varepsilon} \frac{1}{n} D_{\max,\mathcal{F}}(\rho') = D^{\infty}_{\mathcal{F}}(\rho)$$

$$(\sigma_n)_n \ \text{ s.t. } \ D_{\max,\mathcal{F}}(\rho'_n) = D_{\max}(\rho'_n \| \sigma_n)$$

$$\omega_n := \frac{2^{-\eta}\sigma_n + \rho'_n}{1 + 2^{-\eta}} \qquad \eta = \log(2\varepsilon^{-1} - 1) \qquad \omega_n \underset{\varepsilon}{\approx} \rho'_n$$

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{D}_{\Omega}(\omega_n \| \sigma_n) \ \to \ D^{\infty}_{\mathcal{F}}(\rho)$$

Converse bound

Result II

$$r_{\text{prob}}(\rho \to \omega) \leq \frac{\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho)}{D^{\infty}_{\mathcal{F}}(\omega)}$$

$$\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho) := \lim_{n\to\infty} \frac{1}{n} \mathbb{D}_{\Omega,\mathcal{F}}\left(\rho^{\otimes n}\right)$$

$$D^{\infty}_{\mathcal{F}}(\rho) := \lim_{n\to\infty} \frac{1}{n} D_{\mathcal{F}}(\rho^{\otimes n})$$

Compare with

$$r(\rho \to \omega) \leq \frac{D^{\infty}_{\mathcal{F}}(\rho)}{D^{\infty}_{\mathcal{F}}(\omega)}.$$

Main idea

Result I (asymptotic equipartition property)

$+$

$$\mathbb{D}_{\Omega,\mathcal{F}}\left(\frac{\Lambda(\rho)}{\operatorname{Tr}\Lambda(\rho)}\right) \leq \mathbb{D}_{\Omega,\mathcal{F}}(\rho) \;\; \forall \Lambda \in \mathcal{O}$$

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Achievability

affine resource theory

$$\mathcal{F} = \mathrm{aff}(\mathcal{F})$$

(coherence, athermality, asymmetry, imaginarity…)

Result III

In any affine resource theory,

$$r_{\mathrm{prob}}(\rho \to \omega) = \frac{\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho)}{D^{\infty}_{\mathcal{F}}(\omega)}$$

$$\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho) := \lim_{n\to\infty} \frac{1}{n} \mathbb{D}_{\Omega,\mathcal{F}}(\rho^{\otimes n})$$

$$D^{\infty}_{\mathcal{F}}(\rho) := \lim_{n\to\infty} \frac{1}{n} D_{\mathcal{F}}(\rho^{\otimes n})$$

Main idea

$$\exists \Lambda \in \mathcal{O} \ : \ \frac{\Lambda(\rho)}{\mathrm{Tr}\,\Lambda(\rho)} = \omega \quad \Longleftrightarrow \quad \mathbb{D}_{\Omega,\mathcal{F}}(\rho) \geq \mathbb{D}_{\Omega,\mathcal{F}}(\omega)$$

[Regula, Quantum 6, 817 (2022)]

More achievability

Result IV

In entanglement distillation (i.e. $\omega = |\Phi_+\rangle\langle\Phi_+|$),

$$r_{\text{prob}}(\rho \to \Phi_+) = \mathbb{D}^{\infty}_{\Omega,\text{SEP}}(\rho)$$

$$\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho) := \lim_{n\to\infty} \frac{1}{n} \mathbb{D}_{\Omega,\mathcal{F}}(\rho^{\otimes n})$$

Main idea

$$\exists \Lambda \in \mathcal{O} \ : \ \frac{\Lambda(\rho)}{\operatorname{Tr}\Lambda(\rho)} = \omega \quad \Leftarrow \quad \mathbb{D}_{\Omega,\mathcal{F}}(\rho) \geq \mathbb{D}^{\mathcal{F}}_{\Omega,\mathcal{F}}(\omega)$$

[Regula, Quantum 6, 817 (2022)]

standard robustness of ω = generalised robustness of ω

$$\inf_{\sigma,\sigma'\in\mathcal{F},\lambda\in\mathbb{R}_+} \left\{ \lambda \,\middle|\, \omega = \lambda\sigma - (\lambda-1)\sigma' \right\} \qquad \inf_{\sigma\in\mathcal{F},\lambda\in\mathbb{R}_+} \left\{ \lambda \,\middle|\, \omega \leq \lambda\sigma \right\}$$

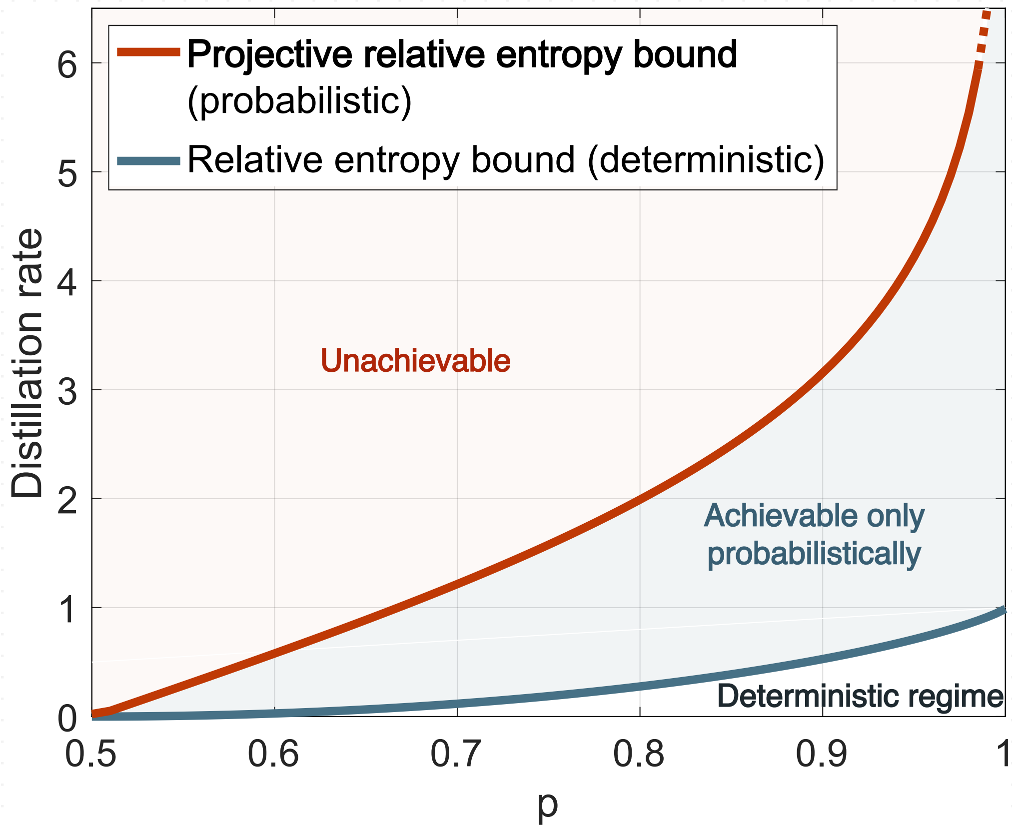
Example: entanglement distillation

$$\rho_p = p\Phi_d + (1-p)\frac{\mathbb{1} - \Phi_d}{d^2 - 1}$$

$$r_{\text{prob}}(\rho \longrightarrow \Phi_2) = \mathbb{D}_{\Omega,\text{SEP}}(\rho_p) = \log \frac{p(d-1)}{1-p}$$

$$> D^{\infty}_{\text{SEP}}(\rho_p) = p \log d + (1-p) \log \frac{d}{d-1} - h_2(p)$$

$$\geq r(\rho_p \longrightarrow \Phi_2)$$



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More insights for distillation

Result V

Every distillation protocol $(\Lambda_n)_n$ which distills at a rate **r** with

error $\varepsilon_n := \frac{1}{2}\left\| \Lambda_n(\rho^{\otimes n}) - \psi^{\otimes \lfloor rn \rfloor} \right\|_1$ satisfies

$$r \le \frac{\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho)}{D^{\infty}_{\min,\mathcal{F}}(\psi)} - \frac{\limsup_{n\to\infty} \frac{1}{n} \log\left(\varepsilon_n^{-1} - 1\right)}{D^{\infty}_{\min,\mathcal{F}}(\psi)}$$

strong converse bound

$$r^{\dagger}_{\text{prob}}(\rho \to \psi) \le \frac{\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho)}{D^{\infty}_{\min,\mathcal{F}}(\psi)}$$

no superexponential error decay

$$\varepsilon_n = 2^{-O(n)}$$

(when $\mathbb{D}^{\infty}_{\Omega,\mathcal{F}}(\rho) < \infty$)

$$\lim_{n\to\infty} \frac{1}{n} \inf\left\{ m \;\middle|\; \frac{\Lambda(\psi^{\otimes m})}{\operatorname{Tr}\Lambda(\psi^{\otimes m})} = \phi^{\otimes n} \right\}$$

[Chitambar et al., PRL 101, 140502 (2008)]
[Vrana and Christandl, CMP 352, 621 (2017)]

Conclusions

- General constraints that **all probabilistic protocols** must satisfy
- In many cases, the **best bounds possible**: achievable for all affine resources, entanglement distillation
- Strictly larger than relative entropy bounds: beyond Shannon theory

Follow-up question

Quantum relative entropy = error exponent of asymmetric hypothesis testing (quantum Stein's lemma)

Projective relative entropy = error exponent in postselected hypothesis testing

[arXiv:2209.10550]

Open problem

- Better understanding of pure-to-pure transformations (e.g. LOCC)

Thank you

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