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Bounding Quantum Capacities via Partial Orders and Complementarity

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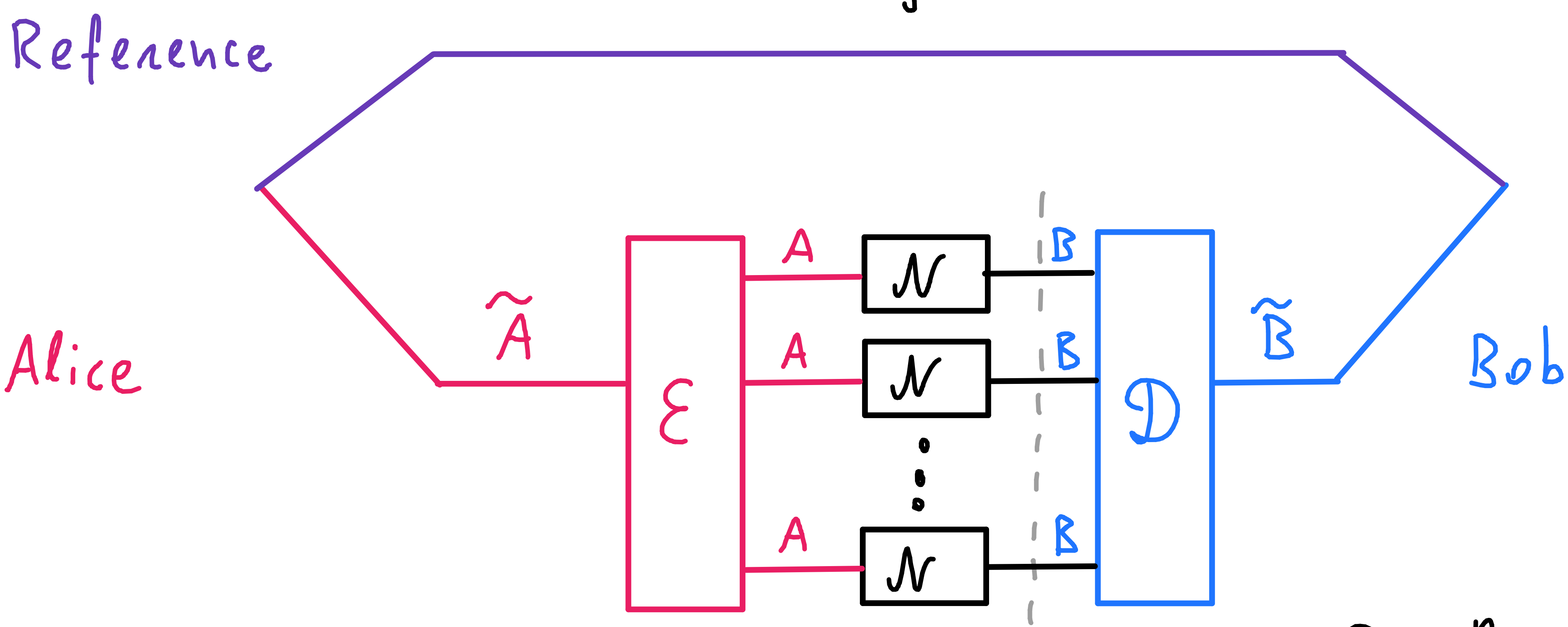
QUANTUM CHANNEL CAPACITIES

Quantum channel $\mathcal{N}: A \to B \longleftrightarrow$ noisy communication link

Quantum capacity $Q(\mathcal{N})$ = highest achievable rate of faithful entanglement transmission

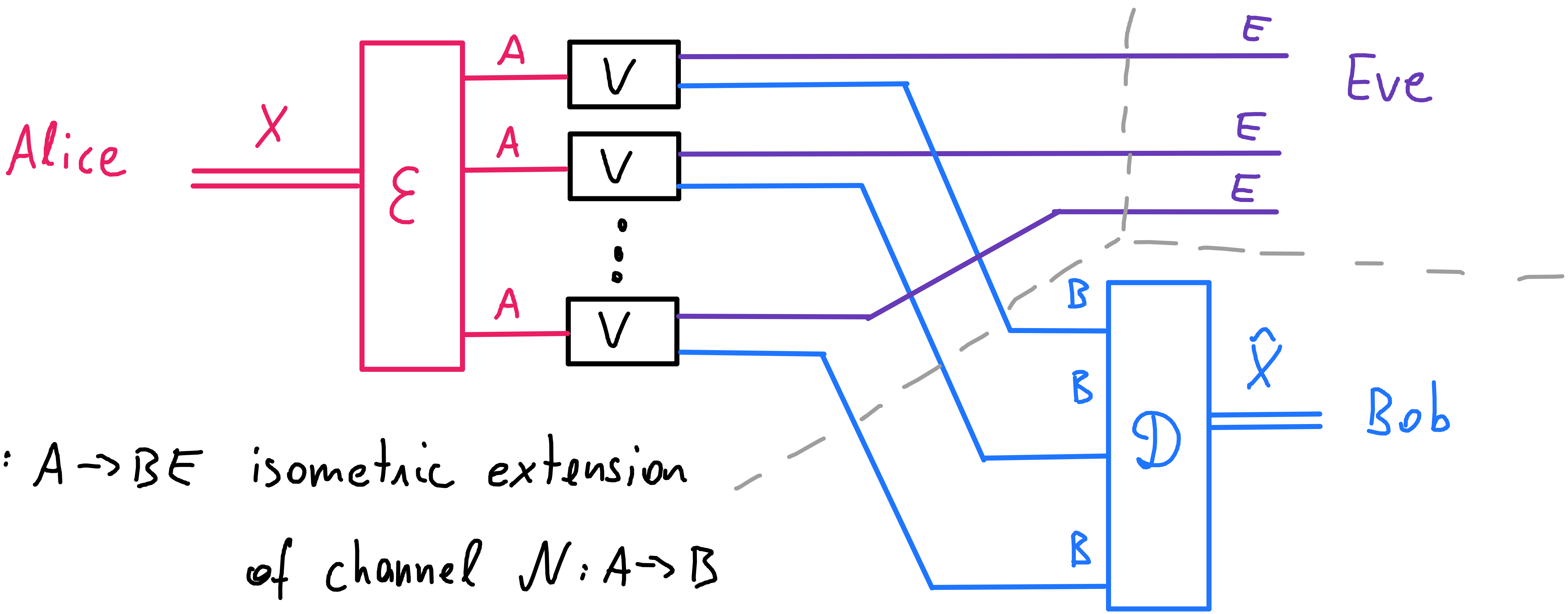


encoder $\mathcal{E}: \tilde{A} \to A^n$ decoder $\mathcal{D}: B^n \to \tilde{B}$

QUANTUM CHANNEL CAPACITIES

Quantum channel $\mathcal{N}: A \to B$ $\longleftrightarrow$ noisy communication link

Private capacity $Q(\mathcal{N})$ = highest achievable rate of secure classical information transmission



$V: A \to BE$ isometric extension of channel $\mathcal{N}: A \to B$

CAPACITIES

Quantum capacity of a quantum channel $\mathcal{N}: A \to B$.

$$Q(\mathcal{N}) = \sup_{n \in \mathbb{N}} \frac{1}{n} Q^{(1)}(\mathcal{N}^{\otimes n})$$

with **coherent information**

$$Q^{(1)}(\mathcal{N}) = \sup_{|\psi\rangle_{AA'}} I(A \rangle B)$$

$$= \sup_{\rho_{XA}} \{ I(X:B) - I(X:E) \}$$

complementary channel

$\mathcal{N}^c : A \to E$

classical-quantum state $\rho_{XA} = \sum_x p_x |x\rangle\langle x|_X \otimes |\psi_x\rangle\langle\psi_x|_A$

$H(A) = -\mathrm{tr}\, \rho_A \log \rho_A$ $\quad$ $I(A \rangle B) = H(B) - H(AB)$ $\quad$ $I(A:B) = H(A) + H(B) - H(AB)$

Capacities

Private capacity of a quantum channel $\mathcal{N}: A \to B$:

$$P(\mathcal{N}) = \sup_{n \in \mathbb{N}} \frac{1}{n} P^{(1)}(\mathcal{N}^{\otimes n})$$

with private information

$$P^{(1)}(\mathcal{N}) = \sup_{\rho_{UA}} \{ I(U:B) - I(U:E) \}$$

classical-quantum state $\rho_{UA} = \sum_u p_u |u\rangle\langle u|_U \otimes \rho_A^u$

Observation: $Q^{(1)}(\mathcal{N}) \leq P^{(1)}(\mathcal{N})$ & $Q(\mathcal{N}) \leq P(\mathcal{N})$

$H(A) = -\mathrm{tr}\, \rho_A \log \rho_A$ $\quad I(A\rangle B) = H(B) - H(AB)$ $\quad I(A:B) = H(A) + H(B) - H(AB)$

WARM-UP

Capacities of a channel $\mathcal{N}: A \to B$ can be bounded by capabilities of the complementary channel $\mathcal{N}^c: A \to E$:

Holevo quantity: $\chi(\mathcal{N}) = \sup_{\rho_{XA}} I(X:B)$

Classical capacity: $C(\mathcal{N}) = \sup_{n \in \mathbb{N}} \frac{1}{n} \chi(\mathcal{N}^{\otimes n})$

entanglement-assisted classical capacity: $C_E(\mathcal{N}) = \sup_{\psi_{AA'}} I(A:B)$

WARM-UP

Capacities of a channel $\mathcal{N}: A \to B$ can be bounded by capabilities of the complementary channel $\mathcal{N}^c: A \to E$:

Holevo quantity: $Q^{(1)}(\mathcal{N}) \leq \chi(\mathcal{N})$

Classical capacity: $Q(\mathcal{N}) \leq C(\mathcal{N})$

entanglement-assisted classical capacity: $2\,Q^{(1)}(\mathcal{N}) \leq C_E(\mathcal{N})$

WARM-UP

Capacities of a channel $\mathcal{N}: A \to B$ can be bounded by capabilities of the complementary channel $\mathcal{N}^c: A \to E$:

Holevo quantity: $Q^{(1)}(\mathcal{N}) \leq \chi(\mathcal{N}) \leq Q^{(1)}(\mathcal{N}) + \chi(\mathcal{N}^c)$

Classical capacity: $Q(\mathcal{N}) \leq C(\mathcal{N}) \leq Q(\mathcal{N}) + C(\mathcal{N}^c)$

entanglement-assisted classical capacity: $2\,Q^{(1)}(\mathcal{N}) \leq C_E(\mathcal{N}) \leq 2\,Q^{(1)}(\mathcal{N}) + C_E(\mathcal{N}^c)$

PARTIAL ORDERS

Degradability:

$$\mathcal{N} \succeq_{deg} \mathcal{M} \quad :\Leftrightarrow \quad \exists \mathcal{D}: \ \mathcal{M} = \mathcal{D} \circ \mathcal{N}$$

$\mathcal{N}$ is called **degradable** $\quad :\Leftrightarrow \quad \mathcal{N} \succeq_{deg} \mathcal{N}^c$

anti-degradable $\quad :\Leftrightarrow \quad \mathcal{N}^c \succeq_{deg} \mathcal{N}$

We know: $\mathcal{N}$ degradable $\Rightarrow P^{(1)}(\mathcal{N}) = P(\mathcal{N}) = Q(\mathcal{N}) = Q^{(1)}(\mathcal{N})$

$\mathcal{N}$ anti-deg. $\Rightarrow P^{(1)}(\mathcal{N}) = P(\mathcal{N}) = Q(\mathcal{N}) = Q^{(1)}(\mathcal{N}) = 0$

[Devetak & Shor '05, Smith '08]

PARTIAL ORDERS

$\mathcal{N} : A \to B$ is called: [Watanabe '12, Cross et al. '15]

less noisy (l.n.) $:\Leftrightarrow P^{(1)}(\mathcal{N}^c) = 0$

$\Leftrightarrow I(U:E) \leq I(U:B) \quad \forall$ c-q states ρ_{UA}

regularized l.n. $:\Leftrightarrow P(\mathcal{N}^c) = 0$

$\Leftrightarrow I(U:E^n) \leq I(U:B^n) \quad \forall n, \rho_{UA^n}$

fully quantum l.n. $:\Leftrightarrow P_E(\mathcal{N}^c) := \sup_{\rho_{AA'}} \{ I(A:E) - I(A:B) \} = 0$

$\Leftrightarrow I(A:E) \leq I(A:B) \quad \forall \rho_{AA'}$

PARTIAL ORDERS

$\mathcal{N}: A \to B$ is called: [Watanabe '12, Cross et al. '15]

more capable (m.c.) $:\Leftrightarrow Q^{(1)}(\mathcal{N}^c) = 0$

$\Leftrightarrow I(X:E) \leq I(X:B) \ \forall$ c-q states ρ_{XA}

regularized m.c. $:\Leftrightarrow Q(\mathcal{N}^c) = 0$

$\Leftrightarrow I(X:E^n) \leq I(X:B^n) \ \forall n, \rho_{XA^n}$

fully quantum m.c. $:\Leftrightarrow Q_q(\mathcal{N}^c) := \sup_{\psi_{AA'}} \{ I(A:E) - I(A:B) \} = 0$

$\Leftrightarrow I(A:E) \leq I(A:B) \ \forall \psi_{AA'}$

Previous Results

Watanabe '12:

$\mathcal{N}$ more capable $\Rightarrow$ $P^{(1)}(\mathcal{N}) = Q^{(1)}(\mathcal{N})$

$\mathcal{N}$ regularized m.c. $\Rightarrow$ $P(\mathcal{N}) = Q(\mathcal{N})$

$\mathcal{N}$ reg. less noisy $\Rightarrow$ $P(\mathcal{N}) = Q(\mathcal{N}) = P^{(1)}(\mathcal{N}) = Q^{(1)}(\mathcal{N})$

Sutter et al. '17:

$\mathcal{N}$ is ε-degradable $:\Leftrightarrow$ $\mathcal{N}^c$ is ε-close to $\mathcal{D} \circ \mathcal{N}$

$$\Rightarrow \quad Q^{(1)}(\mathcal{N}) \leq Q(\mathcal{N}) \leq Q^{(1)}(\mathcal{N}) + S_1(\varepsilon, |E|)$$

$$P^{(1)}(\mathcal{N}) \leq P(\mathcal{N}) \leq P^{(1)}(\mathcal{N}) + S_2(\varepsilon, |E|)$$

$$Q^{(1)}(\mathcal{N}) \leq P^{(1)}(\mathcal{N}) \leq Q^{(1)}(\mathcal{N}) + S_3(\varepsilon, |E|)$$

$S_i(\varepsilon, |E|) \xrightarrow{\varepsilon \to 0} 0$

APPROXIMATE PARTIAL ORDERS

We pick the obvious definition:

e.g., $\mathcal{N}$ is ε-more capable $:\Leftrightarrow$ $I(X:E) \leq I(X:B) + \varepsilon \quad \forall \rho_{XA}$

$$\Leftrightarrow Q^{(1)}(\mathcal{N}^c) \leq \varepsilon$$

Hinche et al. '20: For $\mathcal{N}: A \to B$,

- $\mathcal{N}$ ε-more capable $\Rightarrow$ $Q^{(1)}(\mathcal{N}) \leq P^{(1)}(\mathcal{N}) \leq Q^{(1)}(\mathcal{N}) + \varepsilon$
- $\mathcal{N}$ ε-regularized m.c. $\Rightarrow$ $Q(\mathcal{N}) \leq P(\mathcal{N}) \leq Q(\mathcal{N}) + \varepsilon$
- $\mathcal{N}$ ε-fully quantum less noisy $\Rightarrow$ $Q^{(1)}(\mathcal{N}) \leq Q(\mathcal{N}) \leq Q^{(1)}(\mathcal{N}) + \varepsilon$
- $\mathcal{N}$ ε-fully qu. l.n. and ε-reg. m.c.

$$\Rightarrow P^{(1)}(\mathcal{N}) \leq P(\mathcal{N}) \leq P^{(1)}(\mathcal{N}) + \varepsilon$$

Simplified Bounds

Recall: $\mathcal{N}$ is ε-regularized more capable $\iff Q(\mathcal{N}^c) \leq \varepsilon$

$\Rightarrow$ any $\mathcal{N}$ is ε-reg. more capable with $\varepsilon = Q(\mathcal{N}^c)$

$\Rightarrow$ Result:

$$\boxed{\begin{aligned} Q^{(1)}(\mathcal{N}) &\leq P^{(1)}(\mathcal{N}) \leq Q^{(1)}(\mathcal{N}) + Q^{(1)}(\mathcal{N}^c) \\ Q(\mathcal{N}) &\leq P(\mathcal{N}) \leq Q(\mathcal{N}) + Q(\mathcal{N}^c) \\ Q^{(1)}(\mathcal{N}) &\leq Q(\mathcal{N}) \leq Q^{(1)}(\mathcal{N}) + P_E(\mathcal{N}^c) \\ P^{(1)}(\mathcal{N}) &\leq P(\mathcal{N}) \leq P^{(1)}(\mathcal{N}) + P_E(\mathcal{N}^c) + Q(\mathcal{N}^c) \end{aligned}}$$

This can also be proved "directly" using entropic formulas.

Simplified Bounds

Bounds on capacity separation

$$\boxed{\begin{aligned}
&Q^{(1)}(\mathcal{N}) \leq P^{(1)}(\mathcal{N}) \leq Q^{(1)}(\mathcal{N}) + Q^{(1)}(\mathcal{N}^c)\\
&Q(\mathcal{N}) \leq P(\mathcal{N}) \leq Q(\mathcal{N}) + Q(\mathcal{N}^c)\\
&Q^{(1)}(\mathcal{N}) \leq Q(\mathcal{N}) \leq Q^{(1)}(\mathcal{N}) + P_E(\mathcal{N}^c)\\
&P^{(1)}(\mathcal{N}) \leq P(\mathcal{N}) \leq P^{(1)}(\mathcal{N}) + P_E(\mathcal{N}^c) + Q(\mathcal{N}^c)
\end{aligned}}$$

Bounds on superadditivity

Operational bounds on capacities and information quantities:

- New insights such as: $Q(\mathcal{N}) = Q(\mathcal{N}^c) = 0 \implies P(\mathcal{N}) = 0$
- Numerical bounds on private capacity via quantum capacity

PARTIAL ORDERS FOR STATES

Capacity quantities:

one-way distillable key $K_{\to}(\rho_{AB})$

one-way distillable entanglement $D_{\to}(\rho_{AB}) \leq K_{\to}(\rho_{AB})$

Complementary state: $\rho_{AB}^{c} := \mathrm{tr}_B \, \psi_{ABE} = \rho_{AE}$

purification of ρ_{AB}

Partial orders:

ε- more secret: $K_{\to}(\rho_{AB}^{c}) \leq \varepsilon$

ε- more informative: $D_{\to}(\rho_{AB}^{c}) \leq \varepsilon$

Capacity bounds

Main result:

$$\boxed{\begin{aligned}
D^{(1)}_{\rightarrow}(\rho_{AB}) &\le D_{\rightarrow}(\rho_{AB}) \le D^{(1)}(\rho_{AB}) + K_{\rightarrow}(\rho^c_{AB}) \\
K^{(1)}_{\rightarrow}(\rho_{AB}) &\le K_{\rightarrow}(\rho_{AB}) \le K^{(1)}_{\rightarrow}(\rho_{AB}) + 2K_{\rightarrow}(\rho^c_{AB}) \\
D^{(1)}_{\rightarrow}(\rho_{AB}) &\le K^{(1)}_{\rightarrow}(\rho_{AB}) \le D^{(1)}_{\rightarrow}(\rho_{AB}) + D^{(1)}_{\rightarrow}(\rho^c_{AB}) \\
D_{\rightarrow}(\rho_{AB}) &\le K_{\rightarrow}(\rho_{AB}) \le D_{\rightarrow}(\rho_{AB}) + D_{\rightarrow}(\rho^c_{AB})
\end{aligned}}$$

Operational bounds with similar benefits as before.

Open problems

Does there exist a channel $\mathcal{N}: A \to B$ that is

- regularized less noisy; $\big(\Leftrightarrow P(\mathcal{N}^c) = 0$
  $\Leftrightarrow I(U:E^n) \leq I(U:B^n)\ \forall n\ \forall \rho_{UA^n} \big)$
- but NOT degradable? $\big(\exists\, \mathcal{D} : \mathcal{N}^c = \mathcal{D} \circ \mathcal{N} \big)$

Curious connection to superactivation of private capacity:

P can be superactivated $\Rightarrow$ $\mathrm{DEG} \subsetneq \mathrm{LN}_\infty$

$P(\mathcal{N}) = 0 = P(\mathcal{M})$ but $P(\mathcal{N} \otimes \mathcal{M}) > 0$

Can $P(.)$ be superactivated?

Problem: need either two different classes of zero-cap. channels, or one that is not closed under tensor products.

We only know class of antidegradable channels with $P(\mathcal{N}) = 0$.

We had an idea... **bi-PPT channels** $\mathcal{N}$ s.t. $\mathcal{N}$ and $\mathcal{N}^c$ are PPT, then $Q(\mathcal{N}) = 0 = Q(\mathcal{N}^c) \Rightarrow P(\mathcal{N}) = 0$.

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But: Müller-Hermes & Singh '22: bi-PPT $\Rightarrow$ entanglement-breaking

CAN P(.) BE SUPERACTIVATED?

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BUT! we found numerical examples of approximately bi-PPT $\mathcal{N}$ with $P(\mathcal{N}) \geq Q(\mathcal{N}) > 0$ and $Q(\mathcal{N})$, $Q(\mathcal{N}^c)$, $P(\mathcal{N})$ small.

Summary

We derive operationally meaningful bounds on the quantum and private capacities of channels and states in terms of the complementary object.

The proofs rely on approximate partial orders.

Paper: arXiv: 2202.11688 and to appear in IEEE Transactions

THANKS FOR YOUR ATTENTION!