Reliability Functions of Quantum Information Decoupling and Channel Simulation

[1] Ke Li, [1] Yongsheng Yao

BIID 2022

[arxiv : 2111.06343, 2112.04475]

[1] Harbin Institute of Technology
Institute for Advanced Study in Mathmatics
School of Mathmatics

Outline

- Reliability function
- Quantum entropies and information divergences
- Quantum information decoupling (QID)

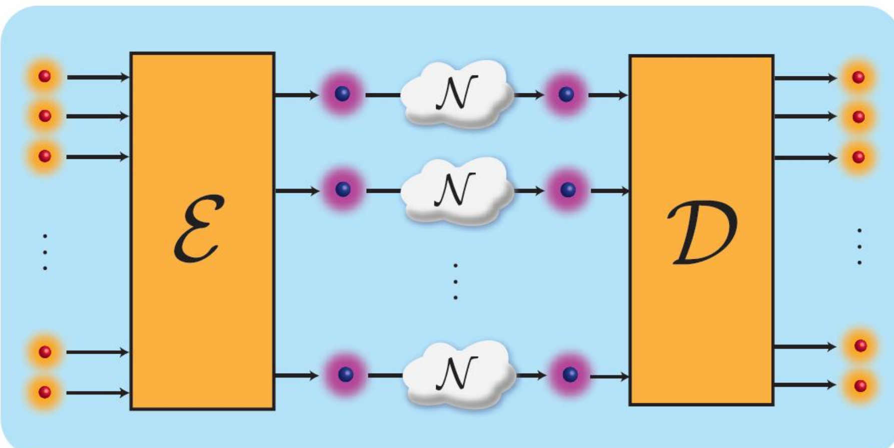
  Ke Li., Yongsheng Yao, arXiv:2111.06343

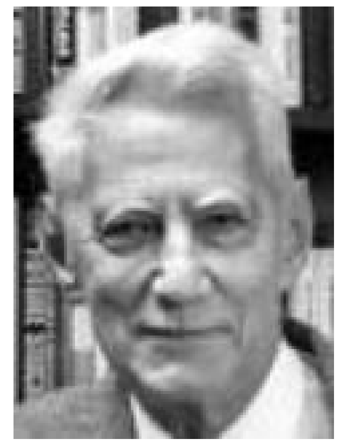
- Application of QID: Quantum channel simulation

  Ke Li, Yongsheng Yao, arXiv:2112.04475

Reliability Function

Shannon's Second Theorem:

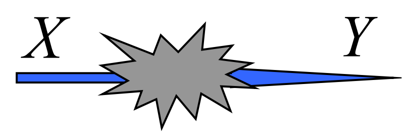




$$Capacity \stackrel{\text{def}}{=} \max\left(\frac{\#\ \ bits\ sent}{\#\ \ channel\ uses}\right)$$

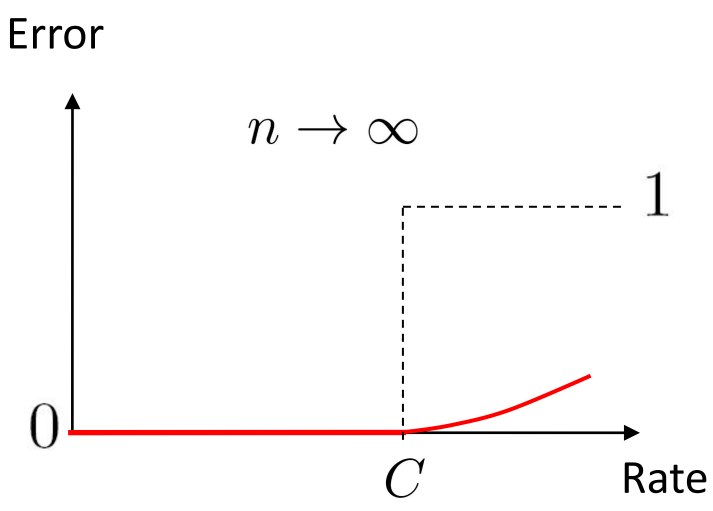
$$C = \max_X I(X:Y),$$

$$I(X:Y) = H(X) + H(Y) - H(XY)$$

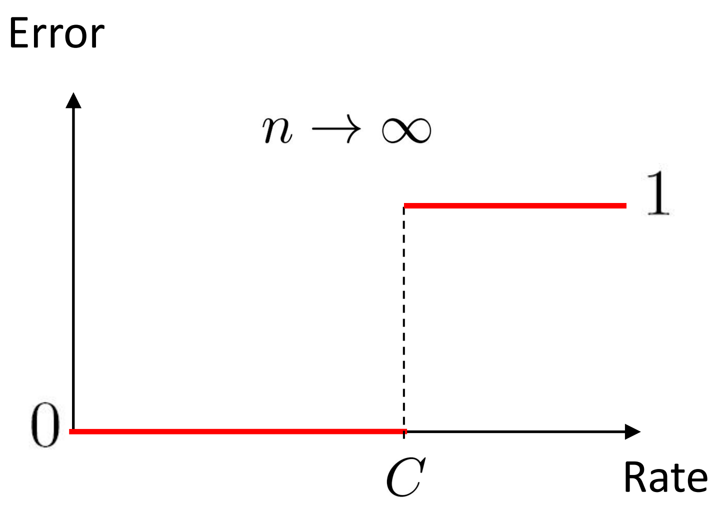


Reliability Function

Shannon's Second Theorem

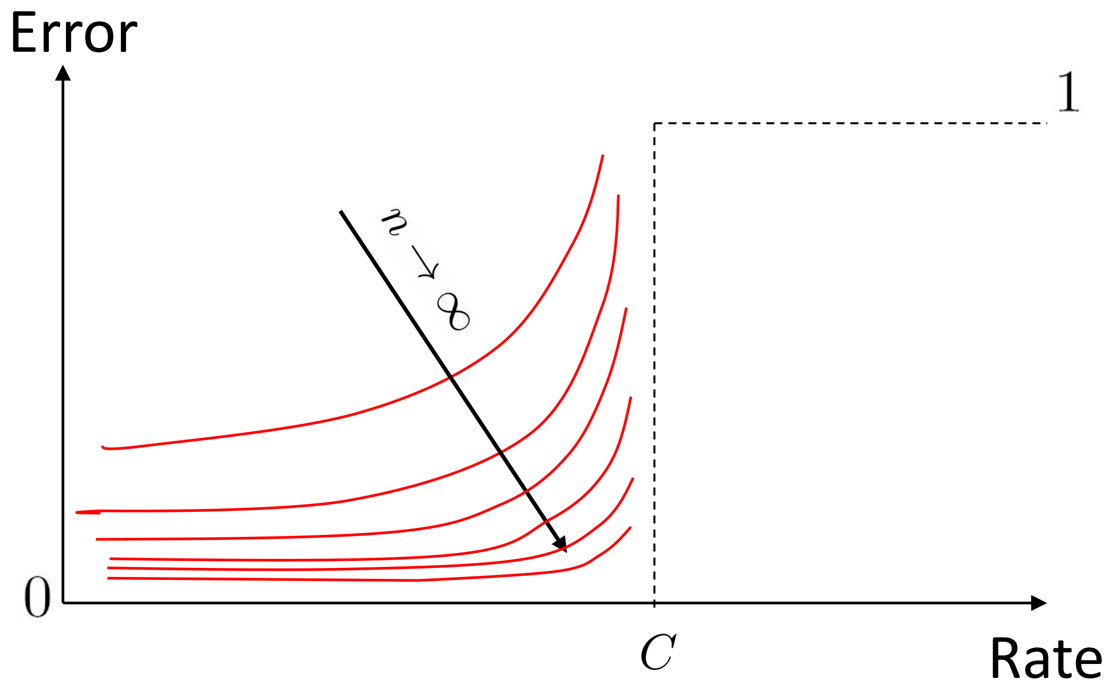


Strong Converse Theorem



Reliability Function

Reliability function of a channel:



$$E(R) \stackrel{\text{def}}{=} -\lim_{n\to\infty} \inf \frac{1}{n} \log P_e(2^{nR}, n), \qquad 0 < R < C$$

Reliability Function

Reliability function of a classical channel:

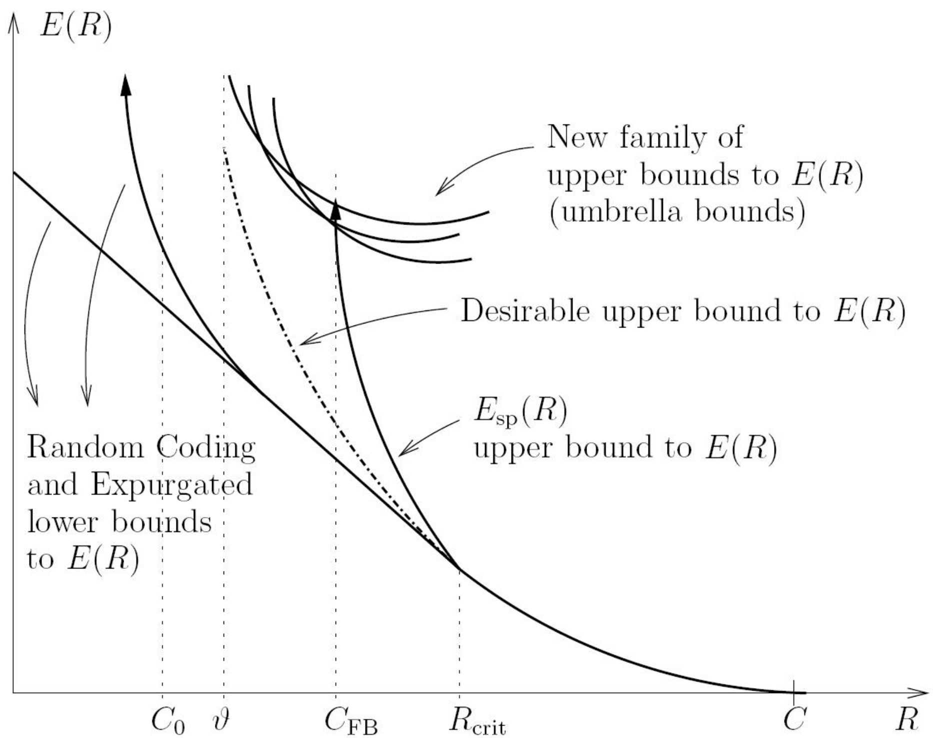


Figure drawn from:

Marco Dalai, IEEE Trans. Inf. Theory, 2013.

Random coding lower bound [Fano'61, Gallager'65]:

$$E_r(R) = \max_{0 \leq \rho \leq 1} [E_0(\rho) - \rho R]$$

Sphere packing upper bound [Shannon-Gallager-Berlekamp'67]:

$$E_{sp}(R) = \sup_{\rho \geq 0} [E_0(\rho) - \rho R]$$

Where

$$E_0(\rho) = \max_P \; -\log \sum_y \left(\sum_x P(x) W_x(y)^{1/(1+\rho)} \right)^{1+\rho}$$

Reliability Function

Reliability functions of quantum information tasks?

No much is known!

For partial results, see:

- Holevo, IEEE Trans. Inf. Theory, 2000.
- Dalai, IEEE Trans. Inf. Theory, 2013.
- Dalai, Winter, IEEE Trans. Inf. Theory, 2017.
- Cheng, Hsieh, Tomamichel, IEEE Trans. Inf. Theory, 2019.
- Cheng, Hanson, Datta, Hsieh, IEEE Trans. Inf. Theory, 2020.

Outline

- Reliability function
- Quantum entropies and information divergences
- Quantum information decoupling (QID)

  K.L., Yongsheng Yao, arXiv:2111.06343
- Application of QID: Quantum channel simulation

  K.L., Yongsheng Yao, arXiv:2112.04475

Quantum entropies and information divergences

Quantum state: positive semidefinite operator with trace 1

von Neumann entropy: $H(A)_\rho := -\operatorname{Tr}(\rho \log \rho)$

Conditional entropy: $H(A|B)_\rho := H(AB)_\rho - H(A)_\rho$

Mutual information: $I(A:B)_\rho := H(A)_\rho - H(AB)_\rho + H(B)_\rho$

Umegaki relative entropy: $D(\rho||\sigma) := \operatorname{Tr}[\rho(\log \rho - \log \sigma)]$

Rényi information divergence:

$$D_\alpha(\rho||\sigma) := \frac{1}{\alpha-1} \log \operatorname{Tr}\left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha$$

$$D'_\alpha(\rho||\sigma) := \frac{1}{\alpha-1} \log \operatorname{Tr}\left(\rho^\alpha \sigma^{1-\alpha}\right)$$

Quantum entropies and information divergences

Quantum state: positive semidefinite operator with trace 1

von Neumann entropy: $H(A)_\rho := -D(\rho||I)$

Conditional entropy: $H(A|B)_\rho := -D(\rho_{AB}||I_A \otimes \rho_B)$

Mutual information: $I(A:B)_\rho := D(\rho_{AB}||\rho_A \otimes \rho_B)$

Umegaki relative entropy: $D(\rho||\sigma) := \mathrm{Tr}[\rho(\log\rho - \log\sigma)]$

Rényi information divergence:

$$D_\alpha(\rho||\sigma) := \frac{1}{\alpha-1}\log \mathrm{Tr}\left(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha$$

$$D'_\alpha(\rho||\sigma) := \frac{1}{\alpha-1}\log \mathrm{Tr}\,(\rho^\alpha\sigma^{1-\alpha})$$

Quantum entropies and information divergences

Quantum state： positive semidefinite operator with trace 1

von Neumann entropy： $H(A)_\rho := -D(\rho||I)$

Conditional entropy: $H(A|B)_\rho := -D(\rho_{AB}||I_A \otimes \rho_B)$

Mutual information: $I(A:B)_\rho := D(\rho_{AB}||\rho_A \otimes \rho_B)$

Umegaki relative entropy： $D(\rho||\sigma) := \mathrm{Tr}[\rho(\log\rho - \log\sigma)]$

Rényi information divergence:

$$D_\alpha(\rho||\sigma) := \frac{1}{\alpha-1}\log \mathrm{Tr}\left(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha$$

$$D'_\alpha(\rho||\sigma) := \frac{1}{\alpha-1}\log \mathrm{Tr}\,(\rho^\alpha\sigma^{1-\alpha})$$

Muller-Lennert, Dupuis, Szehr, Fehr, Tomamichel. JMP, 2013.
Wilde, Winter, Yang. CMP, 2014.

Quantum entropies and information divergences

Sandwiched Rényi Divergence

$$D_\alpha(\rho||\sigma) = \frac{1}{\alpha - 1}\log \mathrm{Tr}\left(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha$$

Sandwiched Rényi
Mutual Information

$$I_\alpha(A:B)_\rho := \min_{\sigma_B} D_\alpha(\rho_{AB}||\rho_A \otimes \sigma_B)$$

Sandwiched Rényi
Mutual Information of
a Quantum Channel

$$I_\alpha(\mathcal{N}_{A\to B}) := \max_{\varphi_{RA}} I_\alpha(R:B)_{id_R \otimes \mathcal{N}(\varphi_{RA})}$$

Outline

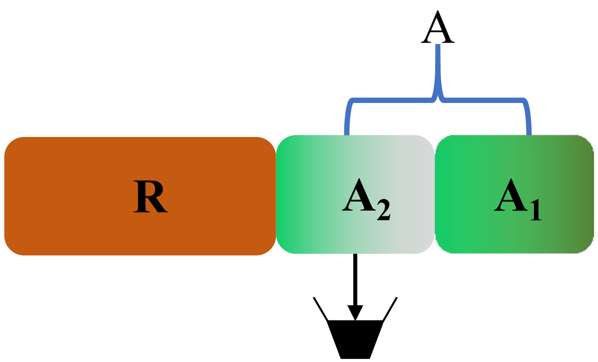
- Reliability function
- Quantum entropies and information divergences
- Quantum information decoupling (QID)

  K.L., Yongsheng Yao, arXiv:2111.06343
- Application of QID: Quantum channel simulation

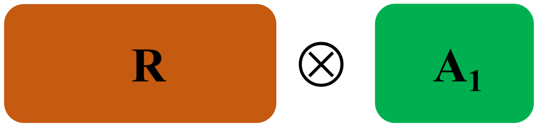
  K.L., Yongsheng Yao, arXiv:2112.04475

Quantum information decoupling

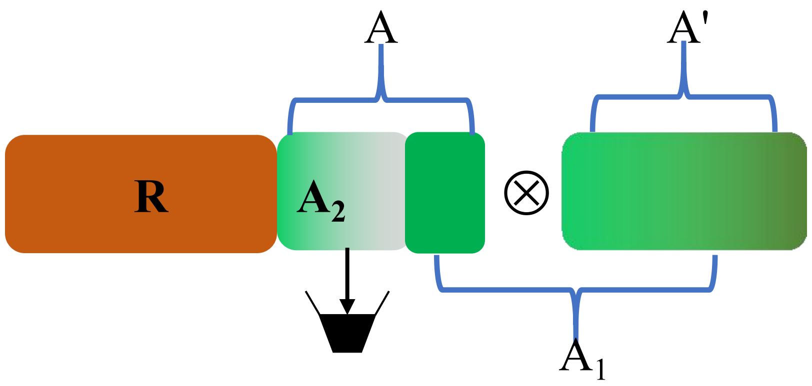
Decoupling



$\mathrm{Tr}_{A_2}(U_A \rho_{RA} U_A^*)$



Catalytic Decoupling



$\mathrm{Tr}_{A_2}(U_{AA'}(\rho_{RA} \otimes \rho_{A'})U_{AA'}^*)$

Horodecki, Oppenheim, Winter. Nature, 2005; CMP, 2007.
Abeyesinghe et al, Proc. R. Soc. A, 2009.

Quantum information decoupling

Applications of decoupling:

- ## Quantum information theory

  Phys. Rev. Lett. 100:230501, 2008; Comm. Math. Phy. 306: 579, 2011; IEEE Tran. Inf. Theory 60:2926, 2014; Phys. Rev. Lett. 121:040504, 2018.

- ## Quantum thermodynamics and statistical physics

  Nature 474:61, 2011; Nat. Commun. 4:1925, 2013; Nat. Phys. 9:721, 2013.

- ## Black hole physics

  J. High Energy Phys. 07:120, 2007; Phys. Rev. Lett. 98:080502, 2007 ; Phys. Rev. Lett. 110:101301, 2013.

Quantum information decoupling

Definition of reliability function

Definition 1 *Let $\rho_{RA} \in \mathcal{S}(RA)$ be a bipartite quantum state. For given $r \geq 0$, we define the optimal performance of decoupling as*

$$P^{\mathrm{dec}}_{R:A}(\rho_{RA}, r) := \min P\big(\operatorname{Tr}_{A_2} U(\rho_{RA} \otimes \sigma_{A'})U^*, \rho_R \otimes \omega_{A_1}\big),$$

where the minimization is over all system dimensions $|A'|$, $|A_1|$, $|A_2|$ such that $|AA'| = |A_1A_2|$ and $\log|A_2| \leq r$, all unitary operations $U : \mathcal{H}_{AA'} \to \mathcal{H}_{A_1A_2}$, and all states $\sigma_{A'} \in \mathcal{S}(A')$, $\omega_{A_1} \in \mathcal{S}(A_2)$.

Definition 2 *Let $\rho_{RA} \in \mathcal{S}(RA)$ be a bipartite quantum state, and $r \geq 0$. The reliability function $E^{\mathrm{dec}}_{R:A}(r)$ of quantum information decoupling for the state ρ_{RA} is defined as*

$$E^{\mathrm{dec}}_{R:A}(r) := \limsup_{n\to\infty} \frac{-1}{n} \log P^{\mathrm{dec}}_{R^n:A^n}\big(\rho_{RA}^{\otimes n}, nr\big).$$

An upper bound for $P^{\text{dec}}_{R:A}(\rho_{RA}, r)$

Lemma .12 (Convex-split lemma). *Let* $\rho_{PQ} \in \mathcal{D}(PQ)$ *and* $\sigma_Q \in \mathcal{D}(Q)$ *be quantum states such that* $\text{supp}(\rho_Q) \subset \text{supp}(\sigma_Q)$. *Let* $k \stackrel{\text{def}}{=} \mathrm{D}_{\max}(\rho_{PQ} \| \rho_P \otimes \sigma_Q)$. *Define the following state*

$$\tau_{PQ_1Q_2\ldots Q_n} \stackrel{\text{def}}{=} \frac{1}{n}\sum_{j=1}^{n} \rho_{PQ_j} \otimes \sigma_{Q_1} \otimes \sigma_{Q_2} \ldots \otimes \sigma_{Q_{j-1}} \otimes \sigma_{Q_{j+1}} \ldots \otimes \sigma_{Q_n} \tag{2}$$

on $n+1$ *registers* $P, Q_1, Q_2, \ldots Q_n$, *where* $\forall j \in [n] : \rho_{PQ_j} = \rho_{PQ}$ *and* $\sigma_{Q_j} = \sigma_Q$. *Then,*

$$\mathrm{D}(\tau_{PQ_1Q_2\ldots Q_n} \| \tau_P \otimes \sigma_{Q_1} \otimes \sigma_{Q_2} \ldots \otimes \sigma_{Q_n}) \leq \log(1 + \frac{2^k}{n}).$$

Anshu, Devabathini, Jain . PRL, 2017.

Lemma 13 *Let* $\rho_{RA} \in \mathcal{S}(RA)$ *and* $\sigma_A \in \mathcal{S}(\mathcal{H}_A)$ *be quantum states such that* $\mathrm{supp}(\rho_A) \subseteq \mathrm{supp}(\sigma_A)$. *Consider the following state*

$$\tau_{RA_1A_2\cdots A_m} := \frac{1}{m}\sum_{i=1}^{m} \rho_{RA_i} \otimes \left[\sigma^{\otimes(m-1)}\right]_{A^m/A_i},$$

where A^m/A_i *denotes the composite system consisting of* $A_1, A_2, \cdots, A_{i-1}, A_{i+1}, \cdots, A_m$ *and* $\left[\sigma^{\otimes(m-1)}\right]_{A^m/A_i}$ *is the product state* $\sigma^{\otimes(m-1)}$ *on these* $m-1$ *systems. Let* $v = v(\rho_R \otimes \sigma_A)$ *denote the number of distinct eigenvalues of* $\rho_R \otimes \sigma_A$. *Then for any* $0 < s \leq 1$,

$$D\left(\tau_{RA_1A_2\cdots A_m} \middle\| \rho_R \otimes (\sigma^{\otimes m})_{A^m}\right) \leq \frac{v^s}{(\ln 2)s} \exp\left\{-(\ln 2)\ s\left(\log m - D_{1+s}(\rho_{RA} \| \rho_R \otimes \sigma_A)\right)\right\}.$$

Ke, Li, Yongsheng Yao, arXiv:2111.06343

A decoupling scheme based on the Lemma

Initial state: ρ_{RA_1}

Catalytic state: $\sigma_{A_2} \otimes \ldots \sigma_{A_m} \otimes \frac{I_{CC'}}{|CC'|}$, $\quad |C| = |C'| = \sqrt{m} \quad m = 2^r$

Unitary operator : $\sum_{i=1}^{m} U_i \otimes V_i \Psi_{CC'} V_i^*$,

where U_i is the swapping between A_1 and A_i, V_i is the Heisenberg-Weyl operator on C and $\Psi_{CC'}$ is the maximally entangled state.

Discarded system: C'

Quantum information decoupling

Main Result:

Proposition *Let* $\rho_{RA} \in \mathcal{S}(RA)$. *For any* $m \in \mathbb{N}$, $0 < s \leq 1$ *and* $\sigma_A \in \mathcal{S}(A)$, *the optimal performance of decoupling* A *from* R *is bounded as*

$$P_{R:A}^{\mathrm{dec}}(\rho_{RA}, r) \leq \sqrt{\frac{v^s}{s}} \exp\left\{ -(\ln 2)\, s\left(r - \frac{1}{2} D_{1+s}(\rho_{RA} \| \rho_R \otimes \sigma_A)\right)\right\},$$

where v *is the number of distinct eigenvalues of* $\rho_R \otimes \sigma_A$.

A lower bound for $P^{\mathrm{dec}}_{R:A}(\rho_{RA}, r)$

Smooth max-information:

$$I^{\delta}_{\max}(A:B)_{\rho} := \min_{\sigma_B \in \mathcal{S}(B)} D^{\delta}_{\max}(\rho_{AB} \| \rho_A \otimes \sigma_B),$$

Smooth quantitite for max-information:

$$\begin{aligned} \delta_{A:B}(\rho_{AB}, \lambda) : &= \min_{\sigma_B} \min\{P(\rho_{AB}, \tilde{\rho}_{AB}) : \tilde{\rho}_{AB} \in \mathcal{S}_{\leq}(AB),\ \tilde{\rho}_{AB} \leq 2^{\lambda} \rho_A \otimes \sigma_B\} \\ &= \min\{\delta : I^{\delta}_{\max}(A:B)_{\rho} \leq \lambda\} \end{aligned}$$

Proposition 19 *Let* $\rho_{RA} \in \mathcal{S}(RA)$. *For* $k \geq 0$, *the optimal performance of decoupling A from R is bounded by the smoothing quantity for the max-information (cf. Eq. (22) in Definition 14):*

$$P^{\mathrm{dec}}_{R:A}(\rho_{RA}, k) \geq \delta_{R:A}(\rho_{RA}, 2k).$$

Ke Li, Yongsheng Yao, arXiv:2111.06343

Theorem 15 *For $\rho_{AB} \in \mathcal{S}(AB)$ and $r \in \mathbb{R}$, we have*

$$\lim_{n\to\infty} \frac{-1}{n} \log \delta_{A^n:B^n}\left(\rho_{AB}^{\otimes n}, nr\right) = \frac{1}{2} \sup_{s\geq 0} \left\{ s\left(r - I_{1+s}(A:B)_\rho\right)\right\}$$

Main methods:

- Gartner-Ellis theorem of large deviation theory

  Dembo, Zeitouni, Springer, 1998.

- Exponent in smoothing the max-relative entropy

  Ke, Li, Yongsheng Yao, Masahito Hayashi, arXiv:2111.01075.

Quantum information decoupling

Main Result:

Theorem *Let $\rho_{RA} \in \mathcal{S}(RA)$ be a bipartite quantum state, and consider the problem of decoupling quantum information A^n from the Reference system R^n in the quantum state $\rho_{RA}^{\otimes n}$. We have*

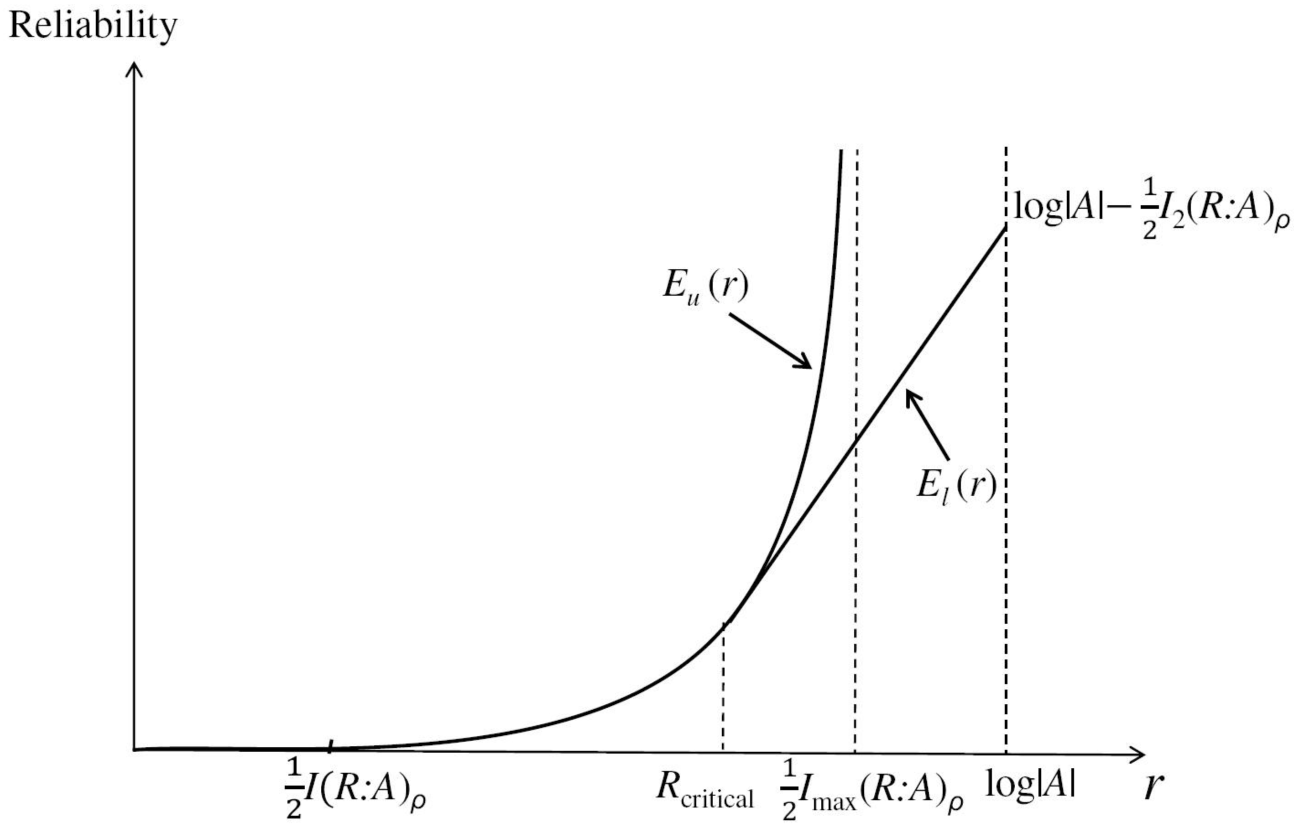
$$E_{R:A}^{\mathrm{dec}}(r) \geq \max_{0 \leq s \leq 1} \left\{ s\Big(r - \frac{1}{2} I_{1+s}(R;A)_\rho\Big) \right\}, \tag{1}$$

$$E_{R:A}^{\mathrm{dec}}(r) \leq \sup_{s \geq 0} \left\{ s\Big(r - \frac{1}{2} I_{1+s}(R;A)_\rho\Big) \right\}. \tag{2}$$

In particular, when $r \leq R_{\mathrm{critical}} := \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} s I_{1+s}(R;A)_\rho \Big|_{s=1}$,

$$E_{R:A}^{\mathrm{dec}}(r) = \max_{0 \leq s \leq 1} \left\{ s\Big(r - \frac{1}{2} I_{1+s}(R;A)_\rho\Big) \right\}. \tag{3}$$

Quantum information decoupling



Outline

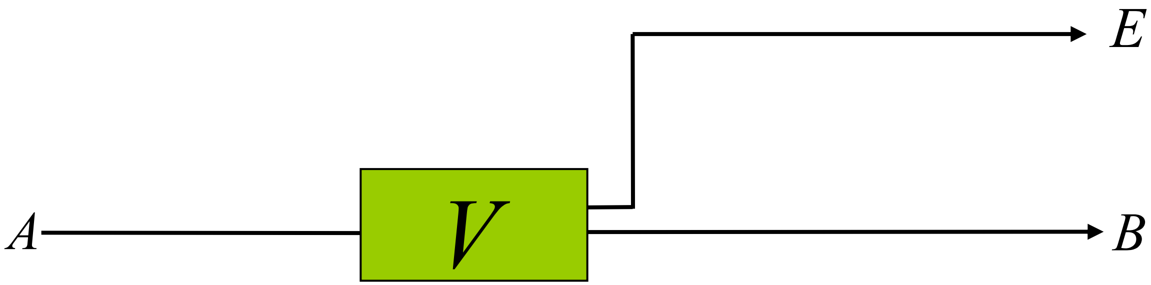
- Reliability function
- Quantum entropies and information divergences
- Quantum information decoupling (QID)

  K.L., Yongsheng Yao, arXiv:2111.06343
- Application of QID: Quantum channel simulation

  K.L., Yongsheng Yao, arXiv:2112.04475

Quantum channel simulation

Quantum channel

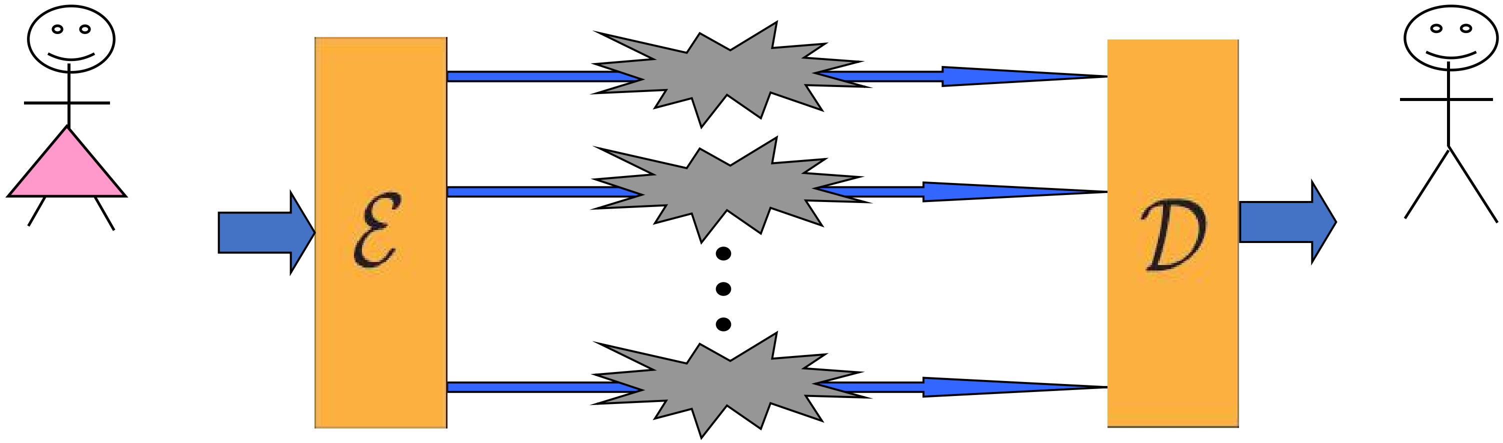


$$\mathcal{N}_{A\to B}(\rho) = \mathrm{Tr}_E\, V\rho V^\dagger$$

with unitary $V : A \to BE$

Quantum channel simulation

Channel capacity



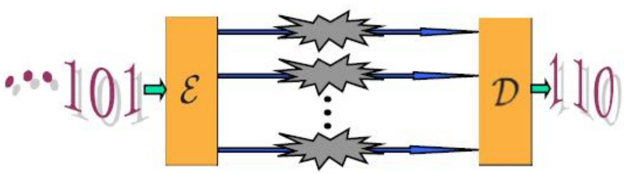
$$Capacity = \max\left(\frac{\#\ \ bits\, sent}{\#\ \ channel\, uses}\right)$$

Quantum channel simulation

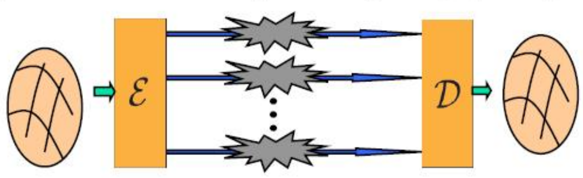
Quantum channel capacities

- Classical capacity $C(\mathcal{N})$

- Private classical capacity $P(\mathcal{N})$



- Quantum capacity $Q(\mathcal{N})$



- When assisted by different auxiliary resources, e.g. free entanglement and classical communications, the capacities behave differently, resulting in a zoo of capacity quantities:

$$C_E, Q_E, C_B, Q_B, C_2, Q_2 \ldots\ldots$$

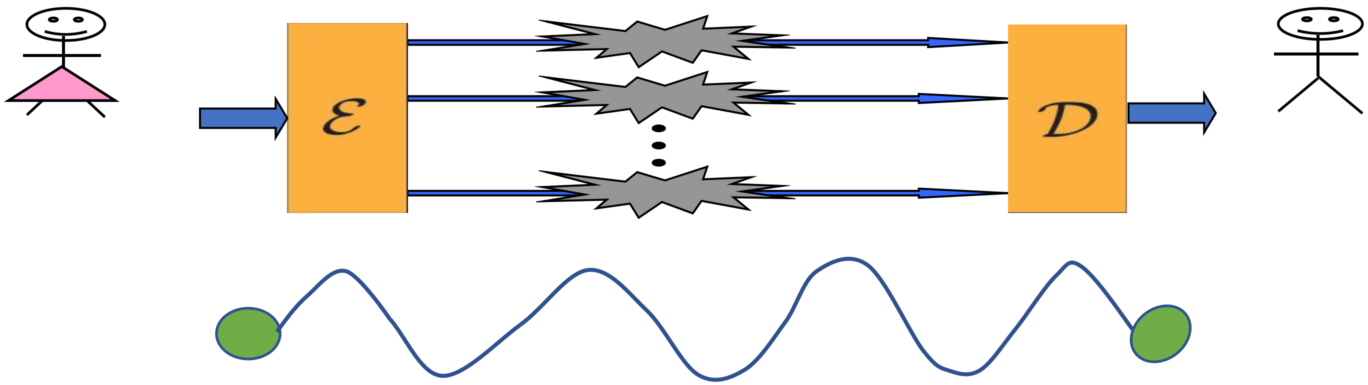
Quantum channel simulation

Entanglement-assisted quantum channel capacities

Bennett-Shor-Smolin-Thapliyal Theorem:

$$C_E(\mathcal{N}) = I(\mathcal{N}) := \max_{\varphi_{RA}} I(R;B)_{\mathrm{id}_R \otimes \mathcal{N}_{A\to B}(\varphi_{RA})}$$

with $I(R;B)_{\mathrm{id}\otimes\mathcal{N}(\varphi_{RA})} = S(\varphi_R) - S(N(\varphi_{RA})) + S(N(\varphi_A))$



Phys. Rev. Lett. 83:3081, 1999; IEEE Tran. Inf. Theory 48:2637, 2002.

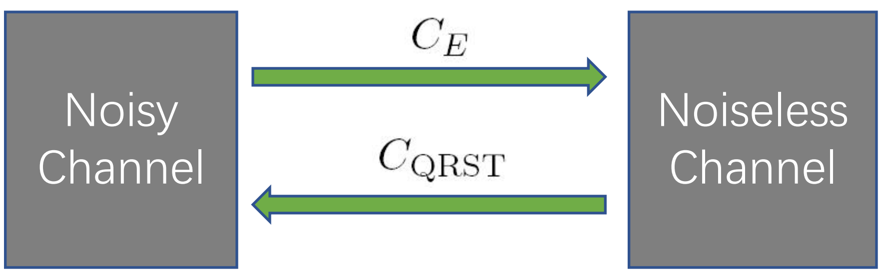
Quantum channel simulation

Entanglement-assisted quantum channel simulation

Quantum Reverse Shannon Theorem (Bennett, Devetak, Harrow, Shor, Winter):

$$C_{\mathrm{QRST}}(\mathcal{N}) = I(\mathcal{N}) := \max_{\varphi_{RA}} I(R;B)_{\mathrm{id}_R \otimes \mathcal{N}_{A\to B}(\varphi_{RA})}$$

with $I(R;B)_{\mathrm{id}\otimes\mathcal{N}(\varphi_{RA})} = S(\varphi_R) - S(N(\varphi_{RA})) + S(N(\varphi_A))$



IEEE Tran. Inf. Theory 60:2926, 2014; Commun. Math. Phys. 306:579, 2011.

Quantum channel simulation

Definition of reliability function

Definition *With the purified distance between two channels* $P(\mathcal{M},\mathcal{N}) := \max_{\varphi_{RA}} P(\mathrm{id}_R \otimes \mathcal{M}(\varphi_{RA}), \mathrm{id}_R \otimes \mathcal{N}(\varphi_{RA}))$, *we define the optimal performance of the reverse Shannon simulation as*

$$P^{\mathrm{sim}}(\mathcal{N}_{A\to B}, c) := \min_{\mathcal{M}_{A\to B}} P(\mathcal{M}_{A\to B}, \mathcal{N}_{A\to B}),$$

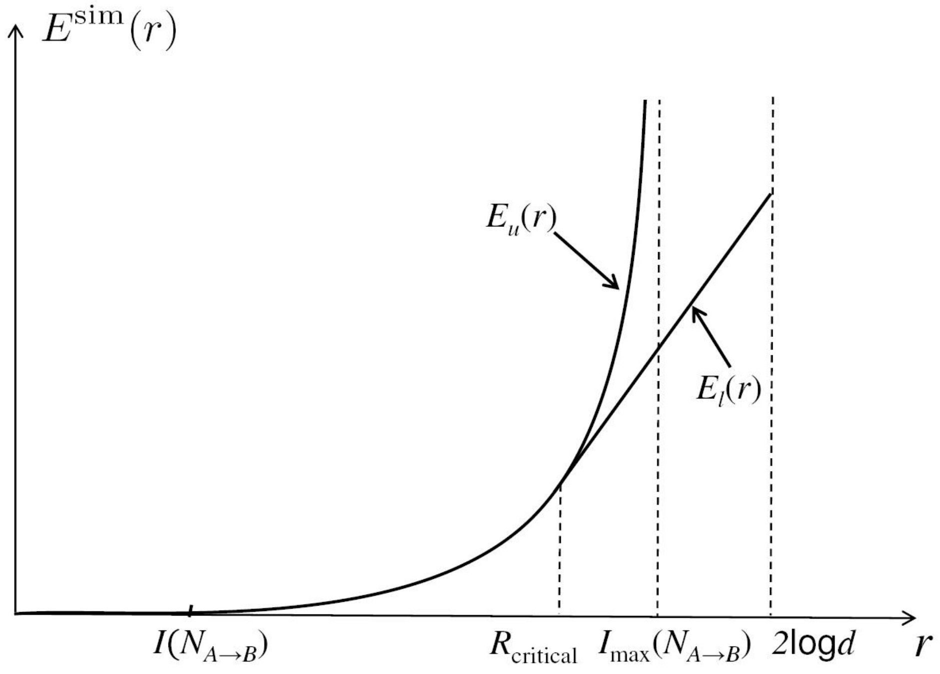
where the minimization is over all the reverse Shannon simulations for $\mathcal{N}_{A\to B}$ *whose classical communication cost does not exceed* c *bits.*

The reliability function of the simulation of $\mathcal{N}_{A\to B}$ *is defined as*

$$E^{\mathrm{sim}}(r) := \limsup_{n\to\infty} \frac{-1}{n} \log P^{\mathrm{sim}}(\mathcal{N}_{A\to B}^{\otimes n}, nr).$$

Quantum channel simulation

Main Result:



$$E_u(r) := \frac{1}{2}\sup_{s\geq 0}\left\{s\big(r - I_{1+s}(\mathcal{N}_{A\to B})\big)\right\}$$

$$E_l(r) := \frac{1}{2}\max_{0\leq s\leq 1}\left\{s\big(r - I_{1+s}(\mathcal{N}_{A\to B})\big)\right\}$$

$$R_{\mathrm{critical}} := \frac{\mathrm{d}}{\mathrm{d}s} sI_{1+s}(\mathcal{N}_{A\to B})|_{s=1}$$

Quantum channel simulation

Main methods:

- Quantum de Finetti reduction

  Christandl, Konig, Renner, Phys. Rev. Lett. 102:020504, 2009.

- Quantum information decoupling

  K.L., Yongsheng Yao, arXiv:2111.06343.

- Additivity of channel Rényi mutual information

  Gupta, Wilde, Commun. Math. Phys. 334:867, 2015.

Outline

- Reliability function
- Quantum entropies and information divergences
- Quantum information decoupling (QID)

  K.L., Yongsheng Yao, arXiv:2111.06343
- Application of QID: Quantum channel simulation

  K.L., Yongsheng Yao, arXiv:2112.04475

Summary and open questions

- We started the *"complete" characterization* of the Reliability Functions for quantum information tasks.
- The results are given in terms of the Sandwiched Rényi Divergence, providing it with operational meanings in characterizing how fast the performance of quantum information tasks approach the perfect for the first time. (see also yesterday's talk by Ke Li.)
- Especially, we give an operational meaning to the channel sandwiched Rényi mutual information, justifying its well definition.

➢ Q1: Reliability functions above the critical points?

➢ Q2: Reliability functions of other QI tasks?

➢ Q3: More types of Rényi information divergences?

Thank you !