# Exact solution for the quantum and private capacities of bosonic dephasing channels

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arXiv:2205.05736

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Stiftung/Foundation

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Single harmonic oscillator (a.k.a. single mode):

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Density operator $\rho \longrightarrow$ infinite matrix (in Fock basis)

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} & \rho_{02} & \cdots \\ \rho_{01}^* & \rho_{11} & \rho_{12} & \cdots \\ \rho_{02}^* & \rho_{12}^* & \rho_{22} & \\ \vdots & & & \ddots \end{pmatrix} \tag{1}$$

Quantum channels:

- Model transformations undergone by open quantum systems:

$$\mathscr{N}(\rho_S) := \mathrm{Tr}_E \left[ U_{SE} \left( \rho_S \otimes |0\rangle\langle 0|_E \right) U_{SE}^\dagger \right], \tag{2}$$

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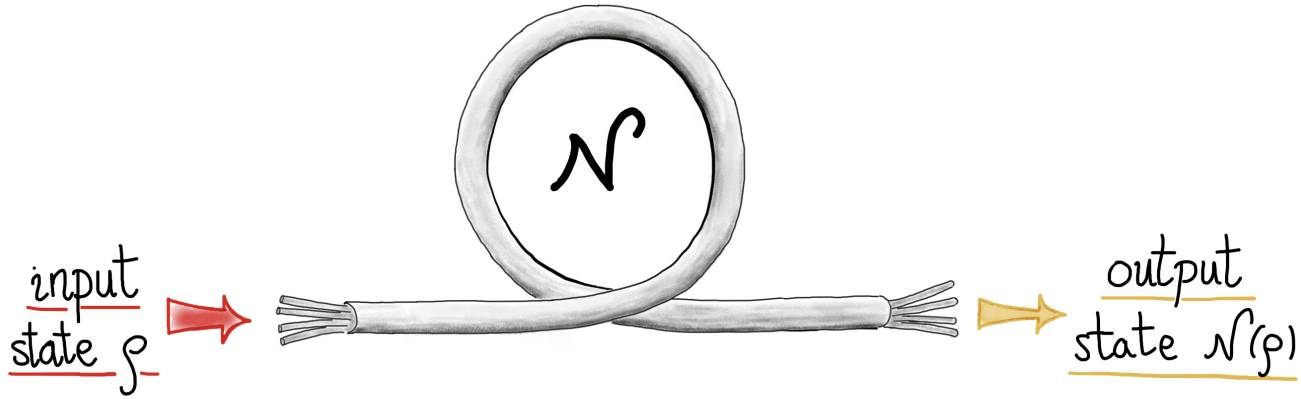

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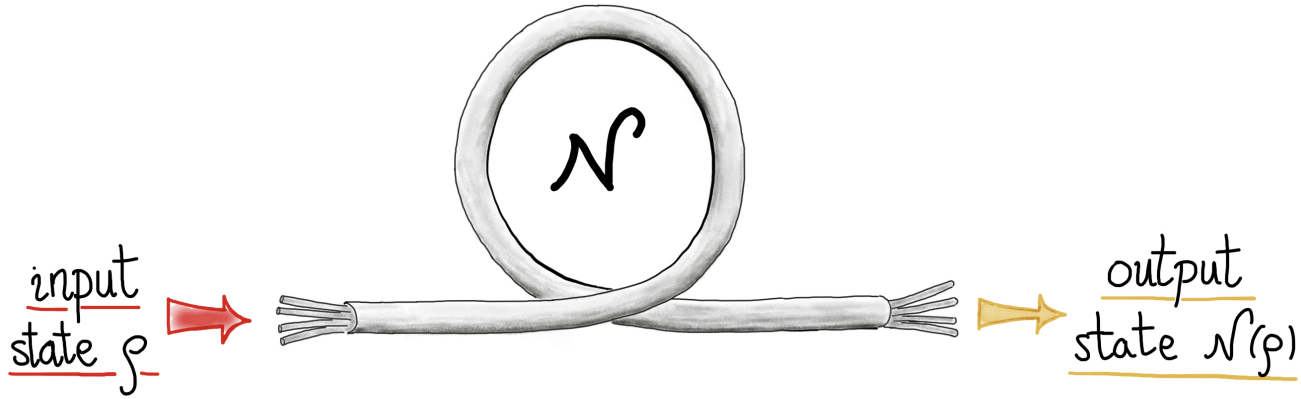


- Kraus representation

$$\mathcal{N}(\rho) = \sum_k M_k \rho M_k^\dagger; \tag{3}$$

  normalisation requires $\sum_k M_k^\dagger M_k = \mathbb{1}$.

# Our model

## Bosonic dephasing channel

Single mode with phase uncertainty:

$$\mathcal{N}_p(\rho) := \int_{-\pi}^{\pi} d\phi \; p(\phi) \; e^{-i\, a^\dagger a\, \phi} \rho \, e^{i\, a^\dagger a\, \phi}, \tag{4}$$

$p : [-\pi, \pi] \to \mathbb{R}_+$ probability distribution, $a^\dagger a$ number operator.

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**Example.** Wrapped Gaussian:[1] $p_\gamma(\phi) := \dfrac{1}{\sqrt{2\pi\gamma}} \displaystyle\sum_{k=-\infty}^{+\infty} e^{-\frac{1}{2\gamma}(\phi+2\pi k)^2}$.

[1] AMM, *Phys. Rev. A* 102:042413, 2020.

Fock basis representation:

$$\mathscr{N}_p(|n\rangle\langle m|) = \int_{-\pi}^{\pi} d\phi \; p(\phi) \; e^{-i\, a^\dagger a\, \phi} \, |n\rangle\langle m| \; e^{i\, a^\dagger a\, \phi}$$

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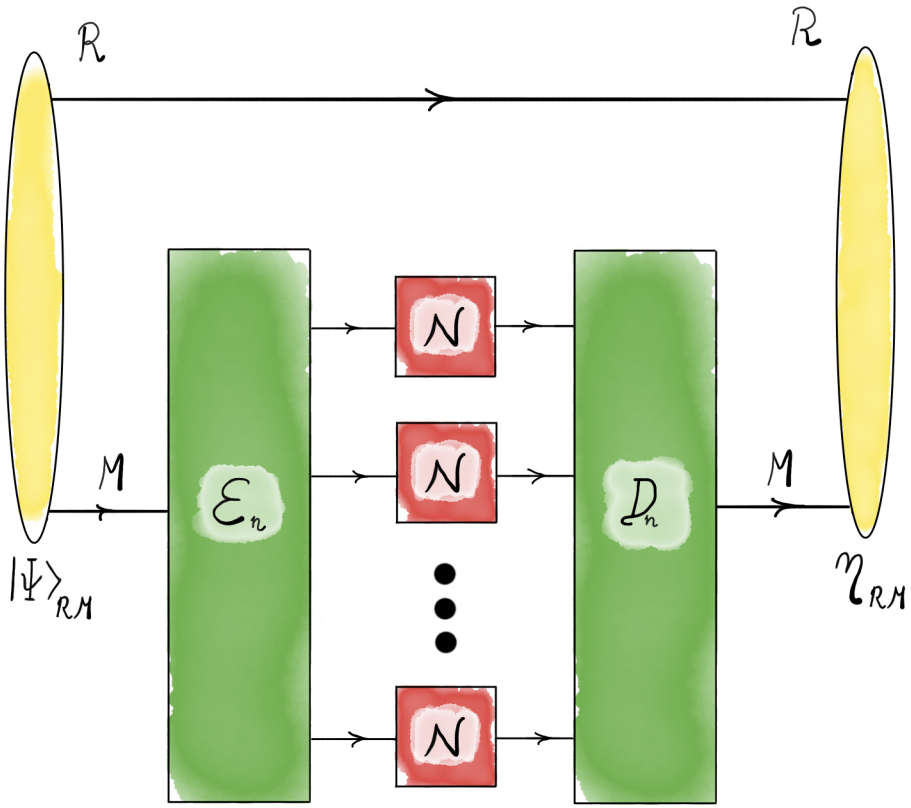

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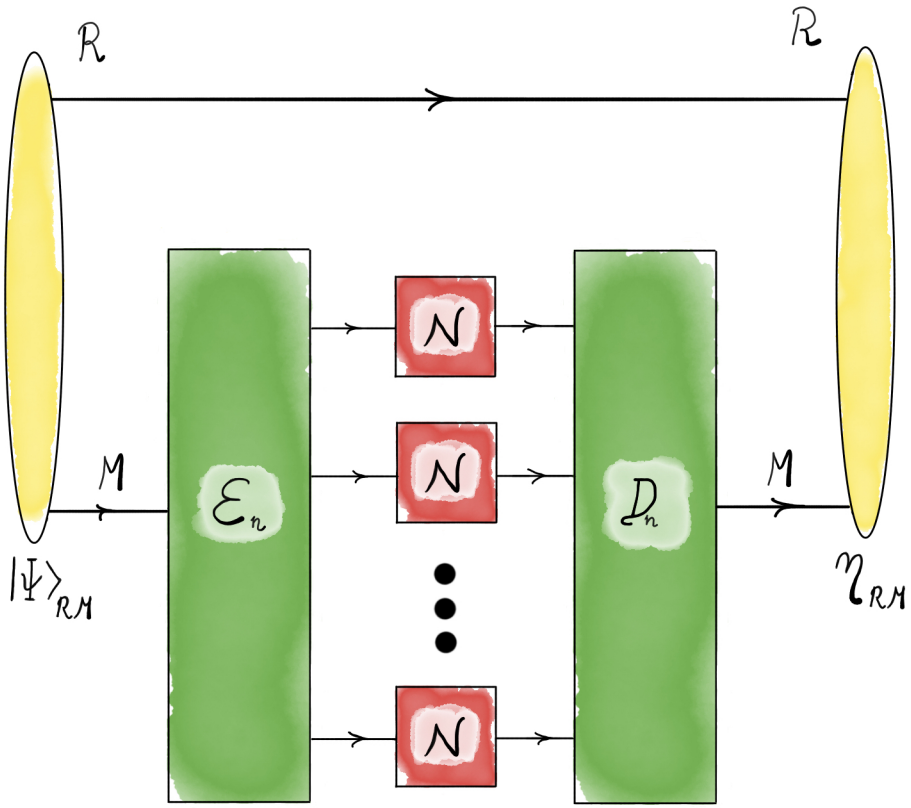


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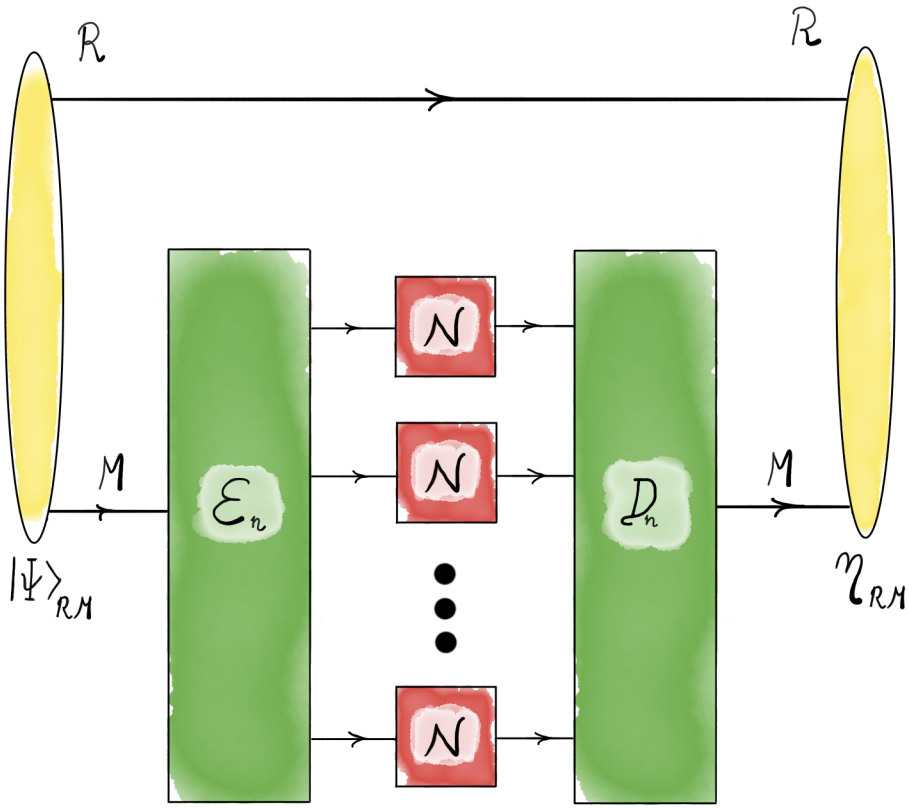


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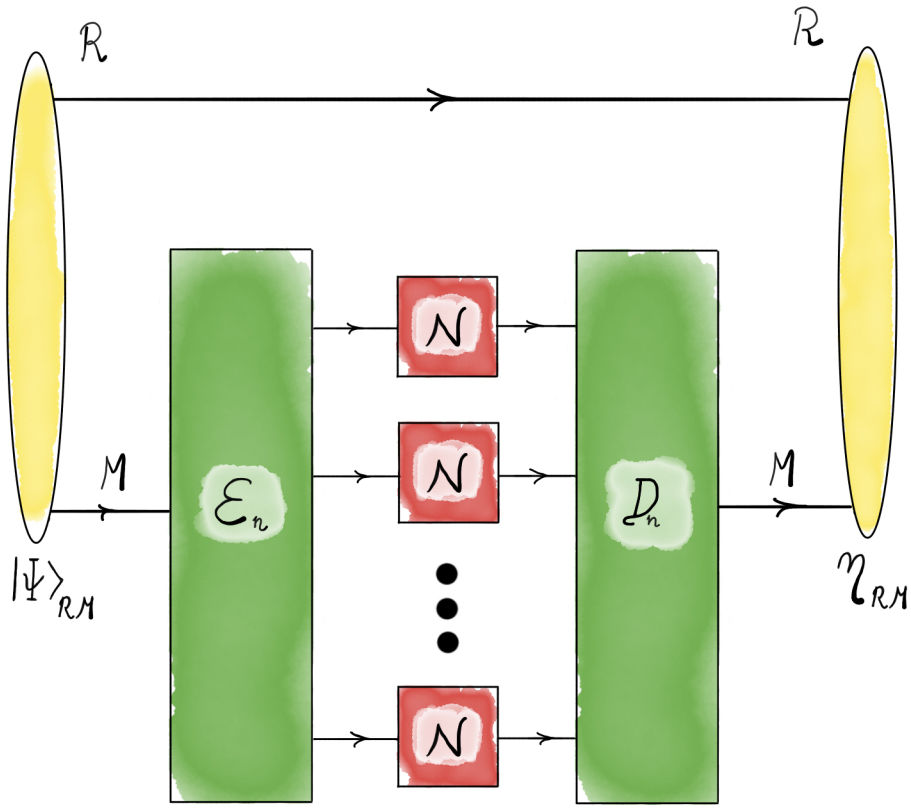


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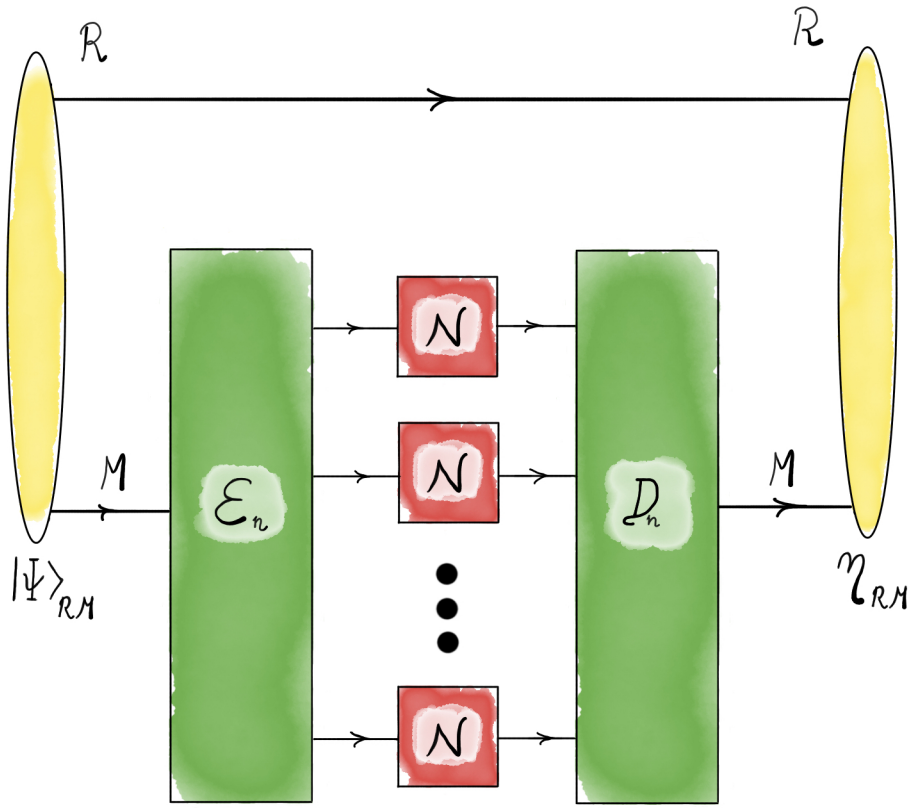


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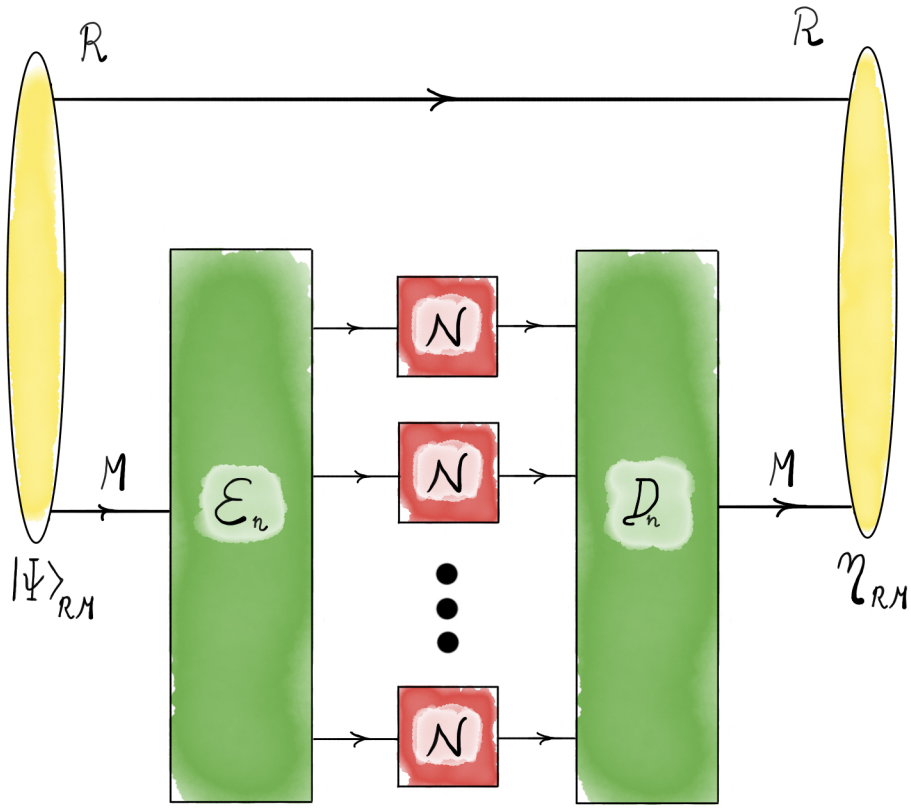


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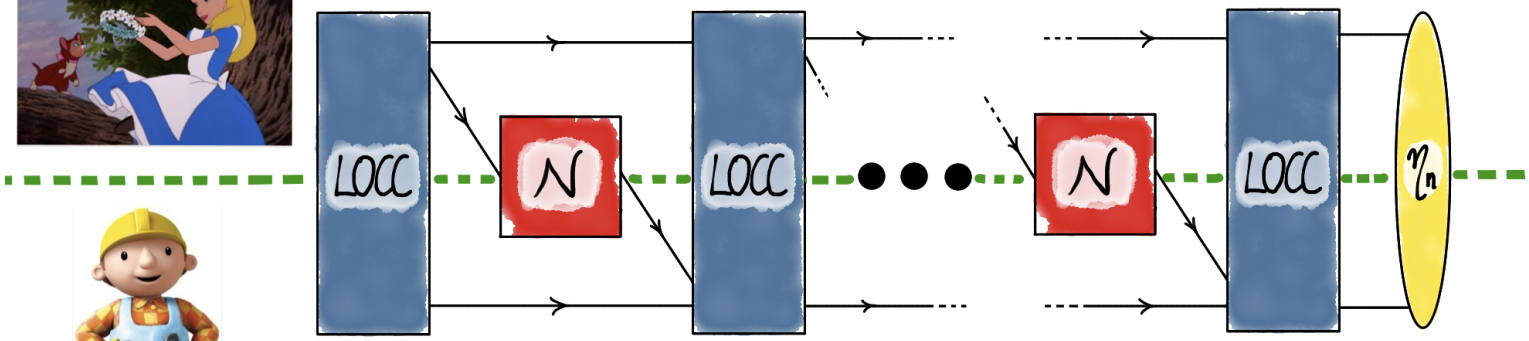

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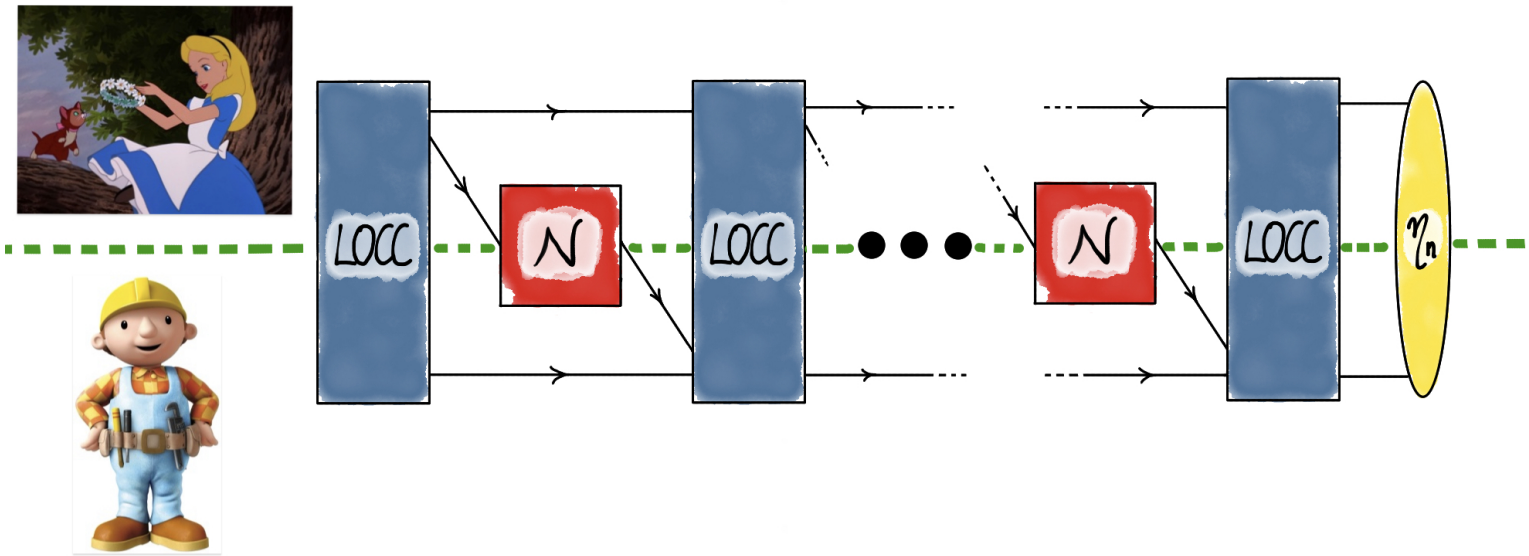


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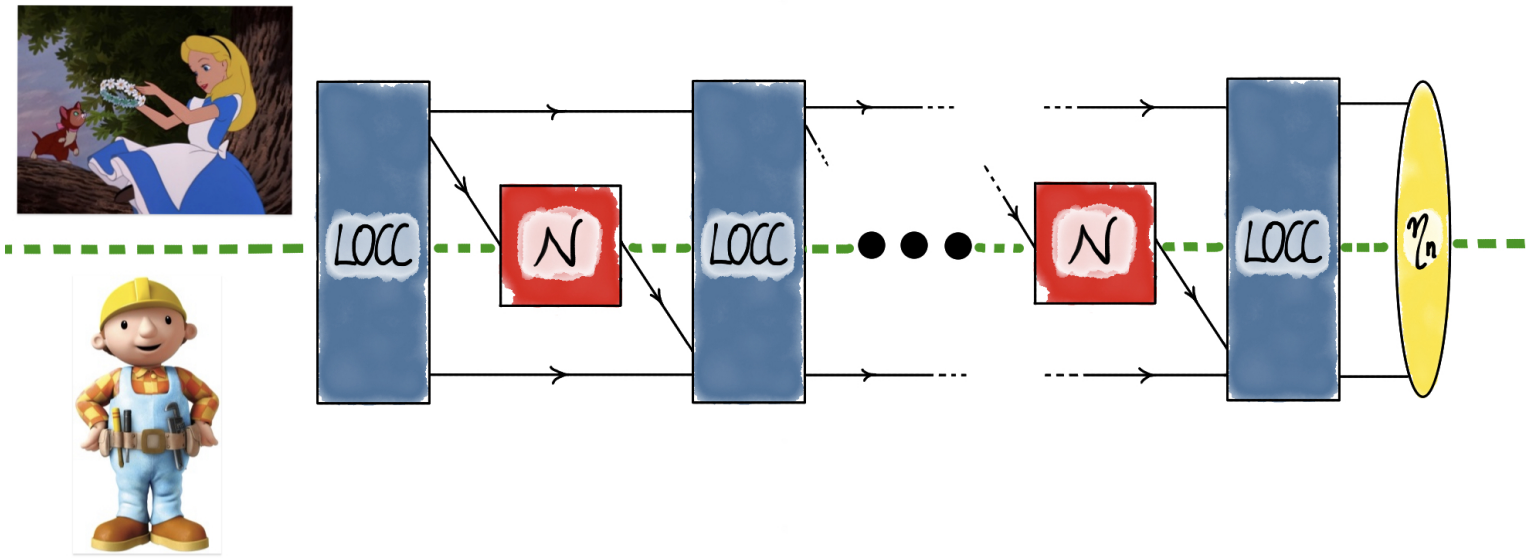


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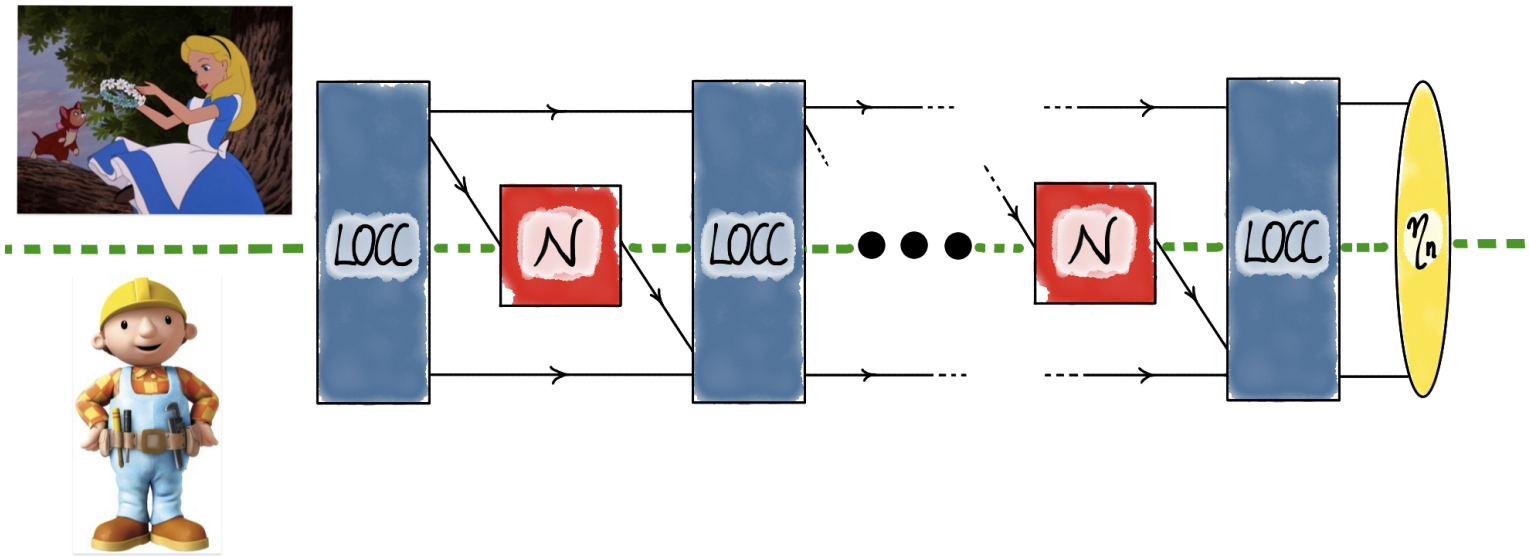


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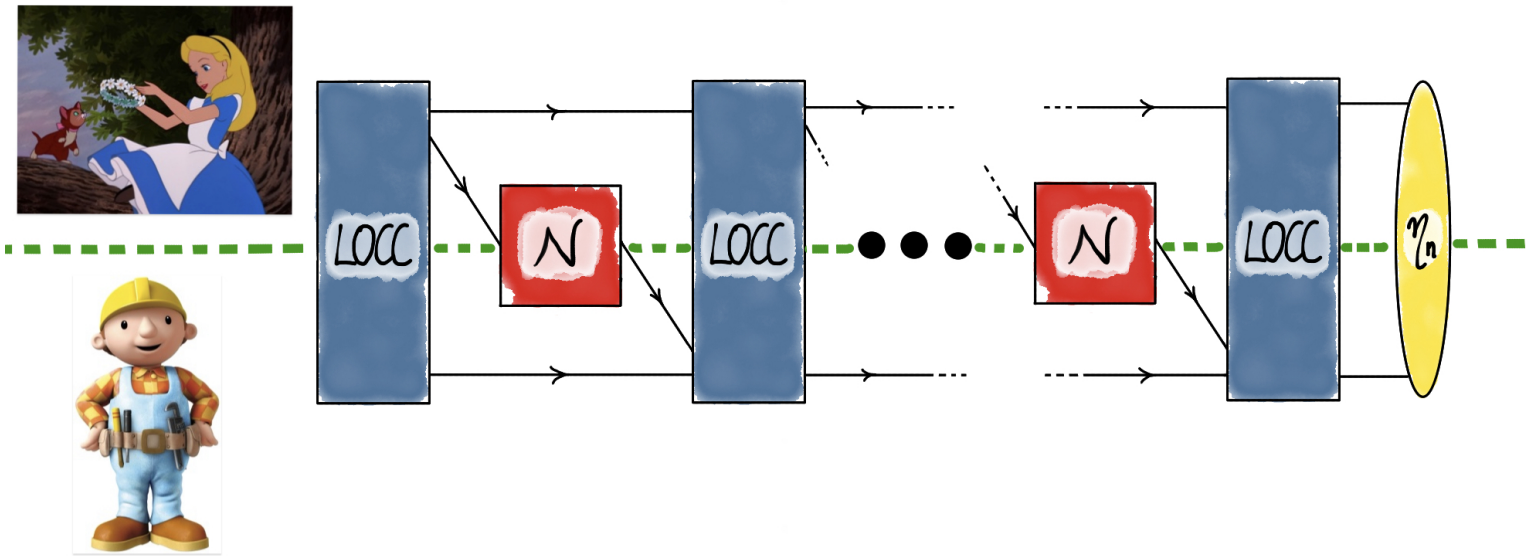


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**Problem**[2]

Compute $\mathscr{C}(\mathcal{N}_p)$, for $\mathscr{C} = Q, Q_{\leftrightarrow}, P, P_{\leftrightarrow}, Q^{\dagger}, Q^{\dagger}_{\leftrightarrow}, P^{\dagger}, P^{\dagger}_{\leftrightarrow}$.

[2] Jiang and Chen, *in Quantum and Nonlinear Optics* 7846:244, 2010. AMM, *Phys. Rev. A* 102:042413, 2020.

# Main result

**Theorem**

*For* $\mathscr{C} = Q, Q_{\leftrightarrow}, P, P_{\leftrightarrow}, Q^{\dagger}, Q^{\dagger}_{\leftrightarrow}, P^{\dagger}, P^{\dagger}_{\leftrightarrow}$ *and any* $p$ *such that*

$$\int_{-\pi}^{+\pi} d\phi \; p(\phi)^{\alpha_0} < \infty \tag{5}$$

*for some* $\alpha_0 > 1$,

$$\mathscr{C}(\mathscr{N}_p) = \log_2(2\pi) - h(p)\,, \tag{6}$$

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- Unlimited classical communication between sender and receiver does not enhance capacities!
- Generating entanglement $\simeq$ generating secret key.
- $\mathscr{C}(\mathscr{N}_p) = D(p\|u)$, where $D(p\|q) := \int d\phi \, p(\phi) \log_2 \frac{p(\phi)}{q(\phi)}$, and $u$ uniform on $[-\pi, \pi]$.

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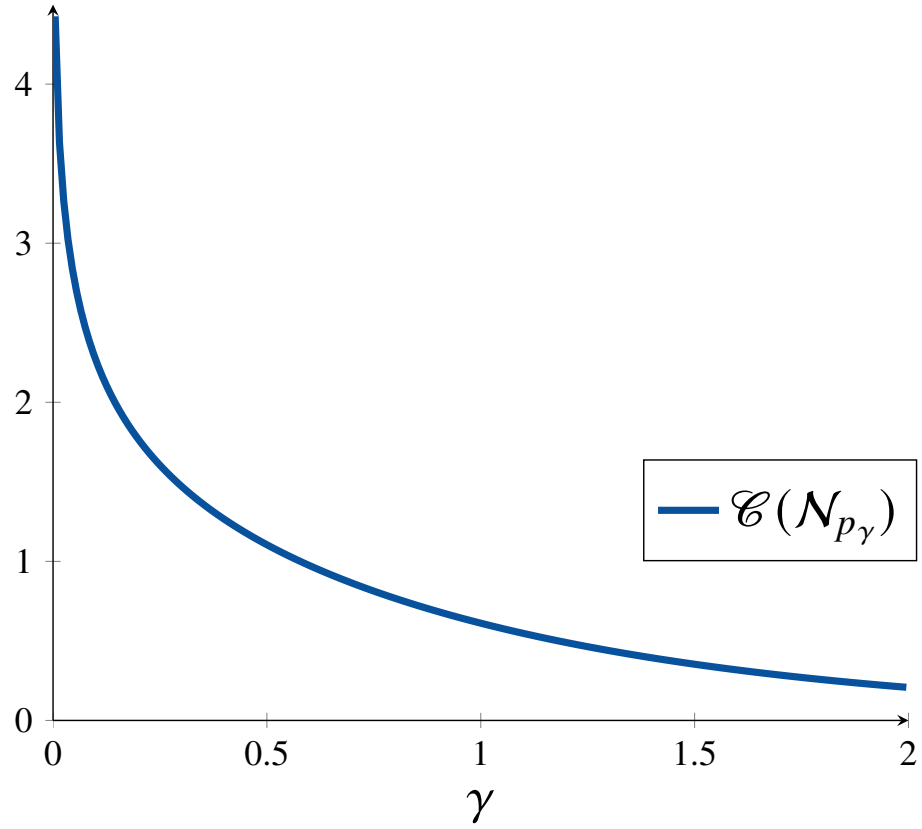
- Our formula generalises PLOB's formula for finite-dim dephasing channels.[3]

[3] PLOB, *Nat. Commun.* 8:15043, 2017.

## Example.

Wrapped Gaussian $\mathcal{N}_{p_\gamma}$:



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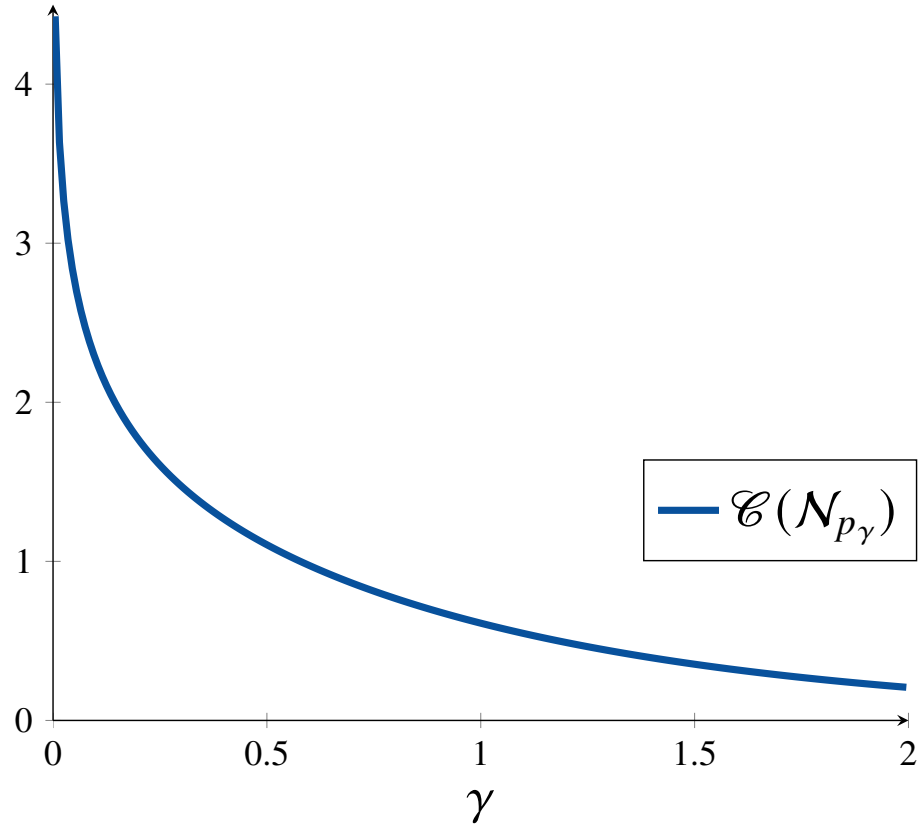


$$\gamma \lesssim 1 \implies \mathscr{C}(\mathcal{N}_{p_\gamma}) \approx \frac{1}{2} \log_2 \frac{2\pi}{e\gamma}$$

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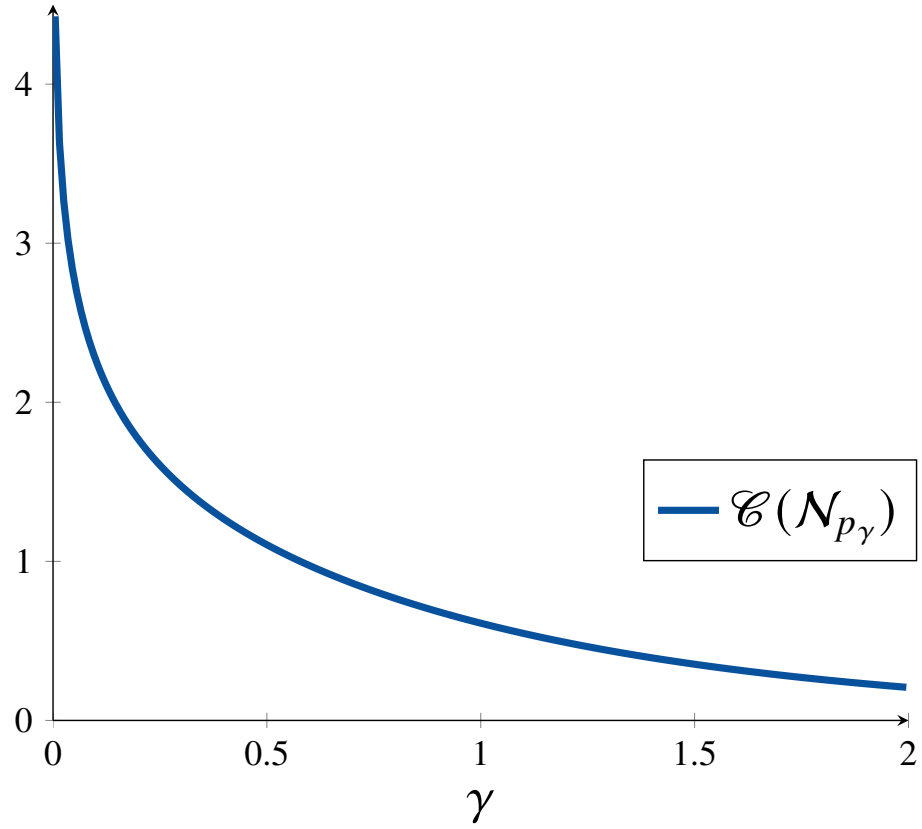


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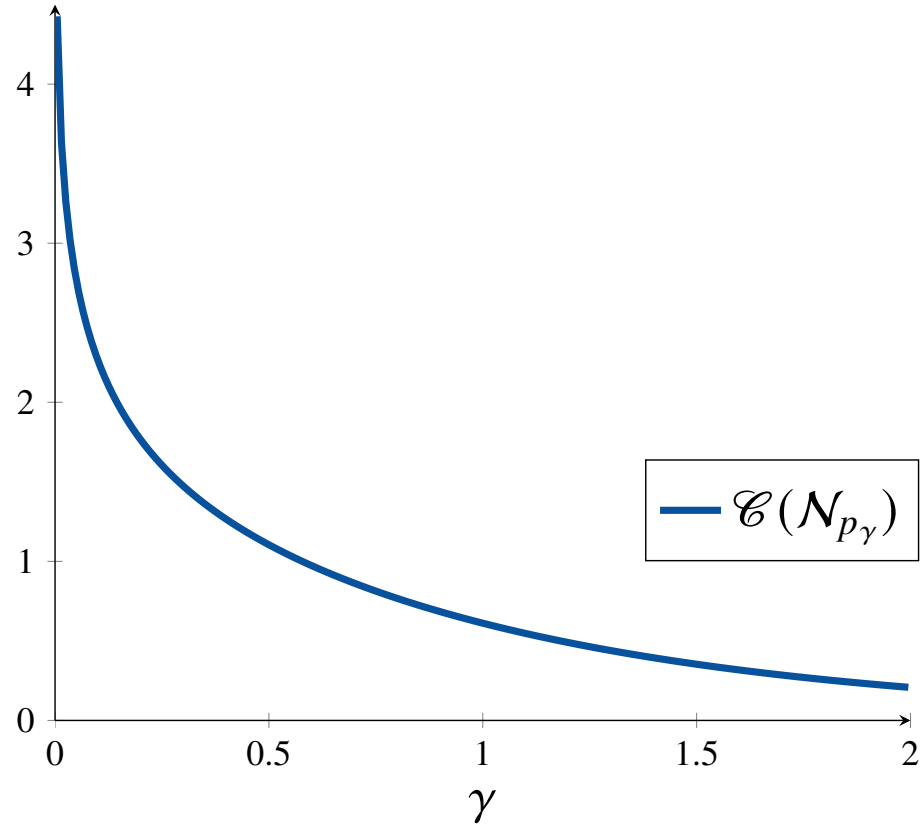


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(a) Lower bound on $Q$ using coherent information (Lloyd–Shor–Devetak theorem):[4]

$$\begin{aligned} Q(\mathcal{N}_p) &\geq \sup_{|\Psi\rangle_{AA'}} I(A\rangle B)_{\mathcal{N}_p^{A'\to B}(\Psi_{AA'})}, \\ I(A\rangle B)_\rho &:= S(\rho_B) - S(\rho_{AB}), \\ S(\omega) &:= -\operatorname{Tr}[\omega \log_2 \omega] \end{aligned} \tag{10}$$

[4] Lloyd, *Phys. Rev. A* 55:1613, 1997. Shor, Lecture notes, 2002. Devetak, *IEEE Trans. Inf. Theory* 51:44, 2005.

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Can we compute the eigenvalues of $T_p^{(d)}$?

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Yes, thanks to Szegő's theorem in asymptotic linear algebra:[5]

$$Q(\mathcal{N}_p) \geq \lim_{d\to\infty} \frac{1}{d} \operatorname{Tr}\left[T_p^{(d)} \log_2 T_p^{(d)}\right] = \log_2(2\pi) - h(p). \tag{12}$$

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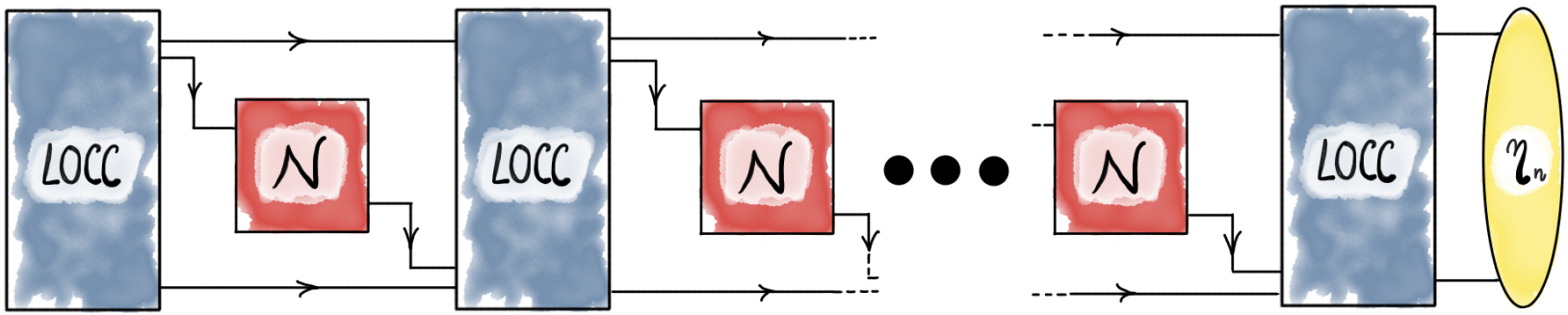

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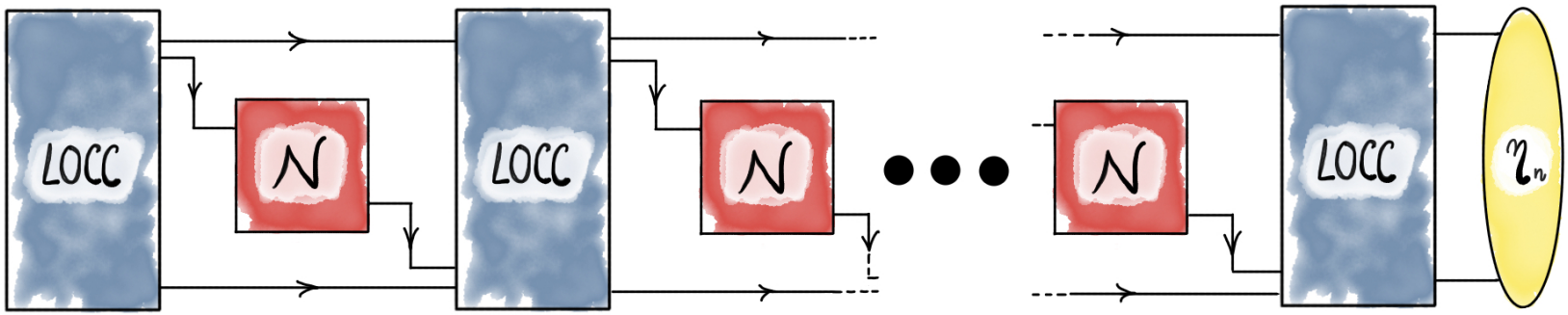


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$\mathcal{N}_p$ is teleportation simulable, i.e.

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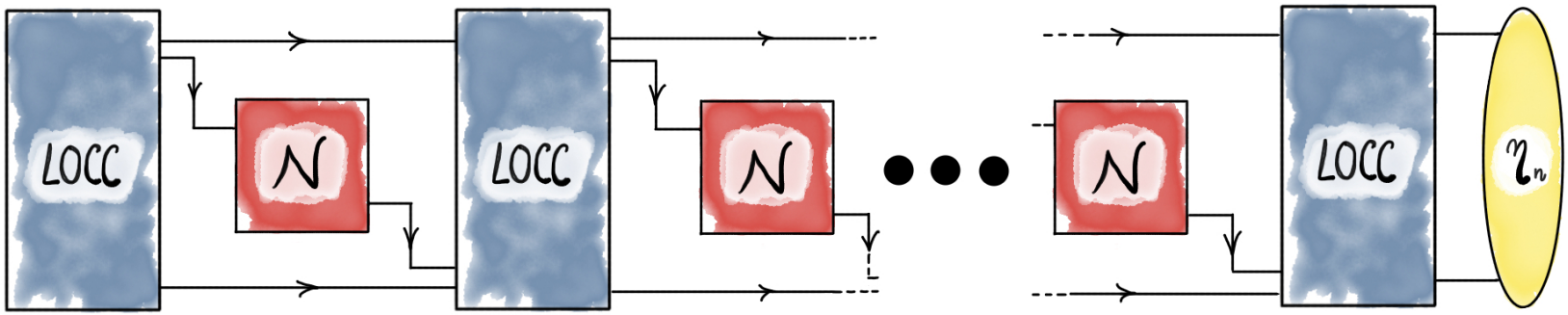


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Whatever Alice and Bob can do on the left, they can do on the right.

Teleportation simulability implies that[6]

$$Q_{\leftrightarrow}(\mathcal{N}_p) \leq \liminf_{d\to\infty} \left( \begin{array}{c} \text{distillable} \\ \text{entanglement} \end{array} \right) (\mathcal{N}_p^{A'\to B}(\Phi_d^{AA'}))$$

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Second appearance of Szegő's theorem!

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