Channel Simulation: Finite Blocklengths and Broadcast Channels

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Channel Simulation with Unconstrained Shared Randomness

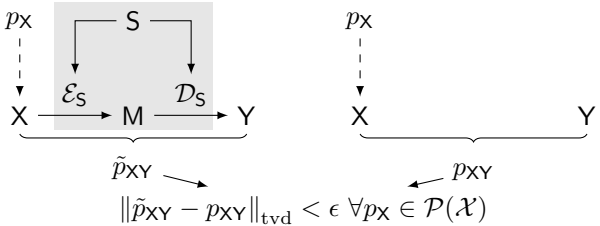

Figure: The task of simulating DMC W with shared randomness S.

- W is some discrete memoryless channel.
- Random variable X is specified by "nature' (*i.e.*, we have no control over it), and is distributed according to some pmf p_{X}.
- S is some random variable shared by both encoder and decoder.
- For each $s \in \mathcal{S}$, encoder $\mathcal{E}_s \in \mathcal{P}(\mathcal{M}|\mathcal{X})$, and decoder $\mathcal{D}_s \in \mathcal{P}(\mathcal{Y}|\mathcal{M})$.

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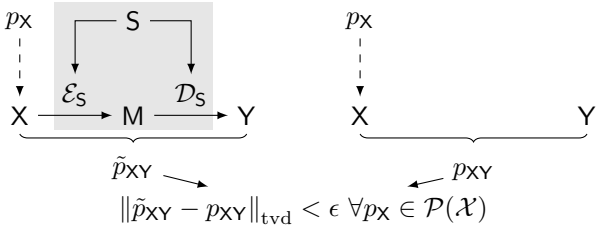

Figure: The task of simulating DMC W with shared randomness S.

- $\widetilde{p}_{\mathsf{XY}}(x, y) := \sum_{s \in \mathcal{S}} p_{\mathsf{S}}(s) \cdot p_{\mathsf{X}}(x) \cdot \sum_{m \in \{1,\ldots,M\}} \mathcal{E}_s(m|x) \cdot \mathcal{D}_s(y|m)$.
- We are interested in finding the smallest M (denoted by $M_\epsilon^\star$) s.t. there exists some $(\mathcal{E}_{\mathsf{S}}, \mathcal{D}_{\mathsf{S}}, \mathsf{S})$ satisfying the ϵ-distance constraint, *i.e.*, $\|p_{\mathsf{XY}} - \widetilde{p}_{\mathsf{XY}}\|_{\mathrm{TVD}} \leqslant \epsilon$ for all p_{X}.

Connections to some other Works

Strong Coordination [CPC10; Cuf13]

Channel simulation can be viewed as *universal* strong coordination of $p_{Y|X}$ for all p_X with the help of unlimited shared randomness. In the scenario with unlimited shared randomness, strong coordination is equivalent to simulating the channel $p_{Y|X}$ for some specific input distribution. Thus,

$$\log M_\epsilon^\star(W_{Y|X}^{\otimes n}) \geqslant \sup_{p_X} \log M_{sc,\epsilon}^\star((p_X \cdot W_{Y|X})^{\otimes n}). \tag{1}$$

Interestingly, the two side of the above inequality match in first order.

One-Shot Bounds on Channel Coding

Channel coding and channel simulation are dual problems.

$$W \xrightarrow{\text{coding}} R_\epsilon \cdot \mathbb{1} \xrightarrow{\text{simulation}} \frac{R_\epsilon}{M_\epsilon} \cdot W \implies \frac{R_\epsilon}{M_\epsilon} \leqslant 1 \text{ when } \epsilon \to 0$$

Connections to some other Works

Reverse Shannon Theorem [BSST02; BDH+14]

The problem has studied in asymptotic setups has been studied in the context of the reverse Shannon theorem. The theorem stipulates that, asymptotically, to faithfully simulate a discrete memoryless channel (DMC) W (or a quantum channel $\mathcal{N}$) with unlimited shared randomness (or shared entanglement), one needs $C(W)$ (or $C_{\mathrm{E}}(\mathcal{N})$) perfect channels per channel simulated.

Channels Conversions

The problem of simulating one channel using another has been studies in asymptotic case (*e.g.*, see [HYAG13]).

Main Results

One-Shot Meta Converse $\log M^{\star}_{\epsilon}(W_{\mathsf{Y}|\mathsf{X}}) \geqslant I^{\epsilon}_{\max}(W_{\mathsf{Y}|\mathsf{X}})$

One-Shot Achievability $\log M^{\star}_{\epsilon}(W_{\mathsf{Y}|\mathsf{X}}) \leqslant I^{\epsilon-\delta}_{\max}(\mathsf{X}:\mathsf{Y})_{W_{\mathsf{Y}|\mathsf{X}}} + \log \frac{4}{\delta^2}$ for any $0 < \delta \leqslant \epsilon$

Third-Order Bounds when $V_{\epsilon} > 0$ $nC(W) + \sqrt{nV_{\epsilon}} \cdot \Phi^{-1}(1-\epsilon) - \log n + O(1) \leqslant \log M^{\star}_{\epsilon}(W^{\otimes n}) \leqslant$
$nC(W) + \sqrt{nV_{\epsilon}} \cdot \Phi^{-1}(1-\epsilon) + \log n + o(1)$

Max-Information of a channel $I^{\epsilon}_{\max}(W_{\mathsf{Y}|\mathsf{X}}) := \inf_{q_{\mathsf{Y}}} \sup_{p_{\mathsf{X}}} D^{\epsilon,\mathsf{X}}_{\max}(p_{\mathsf{X}} \cdot W_{\mathsf{Y}|\mathsf{X}} \| p_{\mathsf{X}} \times q_{\mathsf{Y}})$

Partial smoothed max divergence $D^{\epsilon,\mathsf{X}}_{\max}(p_{\mathsf{XY}} \| q_{\mathsf{XY}}) := \inf_{\substack{\widetilde{p}_{\mathsf{XY}} \in \mathcal{P}(\mathcal{X}\times\mathcal{Y}): \\ \|\widetilde{p}_{\mathsf{XY}} - p_{\mathsf{XY}}\|_{\mathrm{tvd}} \leqslant \epsilon,\, \widetilde{p}_{\mathsf{X}} = p_{\mathsf{X}}}} D_{\max}(\widetilde{p}_{\mathsf{XY}} \| q_{\mathsf{XY}})$

Max-divergence between p and q $D_{\max}(p\|q) := \max_{x\in\mathcal{X}} \log \frac{p(x)}{q(x)}$

Main Results

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One-Shot Achievability $\log M_\epsilon^\star(W_{\mathsf{Y}|\mathsf{X}}) \leqslant I_{\max}^{\epsilon-\delta}(\mathsf{X}:\mathsf{Y})_{W_{\mathsf{Y}|\mathsf{X}}} + \log \frac{4}{\delta^2}$ for any $0 < \delta \leqslant \epsilon$

Second-Order Expansion $\log M_\epsilon^\star(W_{\mathsf{Y}|\mathsf{X}}^{\otimes n}) - n \cdot C(W_{\mathsf{Y}|\mathsf{X}}) - \sqrt{nV_\epsilon(W_{\mathsf{Y}|\mathsf{X}})} \cdot \Phi^{-1}(1-\epsilon) = \Theta(\sqrt{n})$

Third-Order Bounds when $V_\epsilon > 0$ $nC(W) + \sqrt{nV_\epsilon} \cdot \Phi^{-1}(1-\epsilon) - \log n + O(1) \leqslant \log M_\epsilon^\star(W^{\otimes n}) \leqslant$
$nC(W) + \sqrt{nV_\epsilon} \cdot \Phi^{-1}(1-\epsilon) + \log n + o(1)$

ϵ-channel dispersion $V_\epsilon := V_{\max}$ when $\epsilon \leqslant \frac{1}{2}$, and $:= V_{\min}$ when $\epsilon > \frac{1}{2}$

Max-channel dispersion $V_{\max} := \max_{p\in\Pi} \sum_x p(x) \cdot V(p^\star)$

Min-channel dispersion $V_{\min} := \min_{p\in\Pi} \sum_x p(x) \cdot V(p^\star)$

Capacity-achieving distributions $\Pi := \left\{p \in \mathcal{P}(\mathcal{X}) \middle| I(\mathsf{X}:\mathsf{Y})_{p\cdot W_{\mathsf{Y}|\mathsf{X}}} = C(W_{\mathsf{Y}|\mathsf{X}})\right\}$

Channel dispersion w.r.t. the input distribution p $V(p) := \sum_x p(x) \cdot V\left(W_{\mathsf{Y}|\mathsf{X}}(\cdot|x) \middle\| \sum_{\widetilde{x}} p(\widetilde{x}) \cdot W_{\mathsf{Y}|\mathsf{X}}(\cdot|\widetilde{x})\right)$

Divergence variance between p and q $V(p\|q) := \mathrm{Var}\left(\log \frac{p(\mathsf{X})}{q(\mathsf{X})}\right) = \sum_{x\in\mathcal{X}} p(x) \cdot \left(\log \frac{p(x)}{q(x)} - D(p\|q)\right)^2$

Main Results

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One-Shot Achievability $\log M_\epsilon^\star(W_{\mathsf{Y}|\mathsf{X}}) \leqslant I_{\max}^{\epsilon-\delta}(\mathsf{X}:\mathsf{Y})_{W_{\mathsf{Y}|\mathsf{X}}} + \log \frac{4}{\delta^2}$ for any $0 < \delta \leqslant \epsilon$

Second-Order Expansion $\log M_\epsilon^\star(W_{\mathsf{Y}|\mathsf{X}}^{\otimes n}) - n \cdot C(W_{\mathsf{Y}|\mathsf{X}}) - \sqrt{nV_\epsilon(W_{\mathsf{Y}|\mathsf{X}})} \cdot \Phi^{-1}(1-\epsilon) = \Theta(\sqrt{n})$

Third-Order Bounds when $V_\epsilon > 0$ $nC(W) + \sqrt{nV_\epsilon} \cdot \Phi^{-1}(1-\epsilon) - \log n + O(1) \leqslant \log M_\epsilon^\star(W^{\otimes n}) \leqslant$
$nC(W) + \sqrt{nV_\epsilon} \cdot \Phi^{-1}(1-\epsilon) + \log n + o(1)$

Moderate-Deviation Expansion $\frac{1}{n}\log M_{\epsilon_n}^\star(W_{\mathsf{Y}|\mathsf{X}}^{\otimes n}) - C(W_{\mathsf{Y}|\mathsf{X}}) - \sqrt{2V_{\max}(W_{\mathsf{Y}|\mathsf{X}})} \cdot a_n = o(a_n)$

$\epsilon_n \leftarrow 2^{-na_n^2}$, and $\{a_n\}_{n\in\mathbb{N}}$ is some *moderate sequence*, *i.e.*, $a_n \searrow 0$ but $na_n^2 \to +\infty$.

Max-channel dispersion $V_{\max} := \max_{p\in\Pi} \sum_x p(x) \cdot V(p^\star)$

Capacity-achieving distributions $\Pi := \left\{p \in \mathcal{P}(\mathcal{X}) \middle| I(\mathsf{X}:\mathsf{Y})_{p\cdot W_{\mathsf{Y}|\mathsf{X}}} = C(W_{\mathsf{Y}|\mathsf{X}})\right\}$

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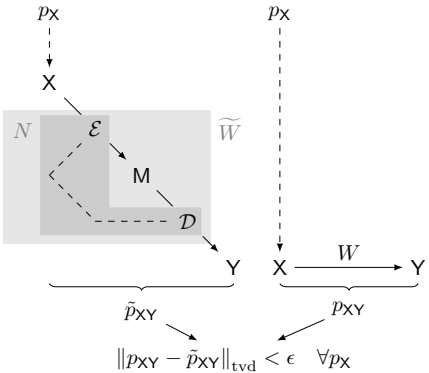
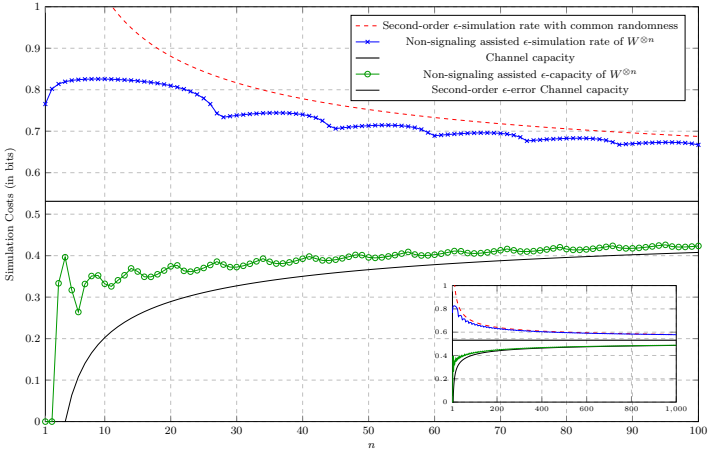
Divergence variance between p and q $V(p\|q) := \mathrm{Var}\left(\log \frac{p(\mathsf{X})}{q(\mathsf{X})}\right) = \sum_{x\in\mathcal{X}} p(x) \cdot \left(\log \frac{p(x)}{q(x)} - D(p\|q)\right)^2$

Main Results

No-Signaling channel simulation $\log C^{\mathrm{NS}}(W,\epsilon) = \left\lceil I^{\epsilon}_{\max}(W_{\mathsf{Y}|\mathsf{X}}) \right\rceil_{\log \mathbb{Z}_{>0}}$ where

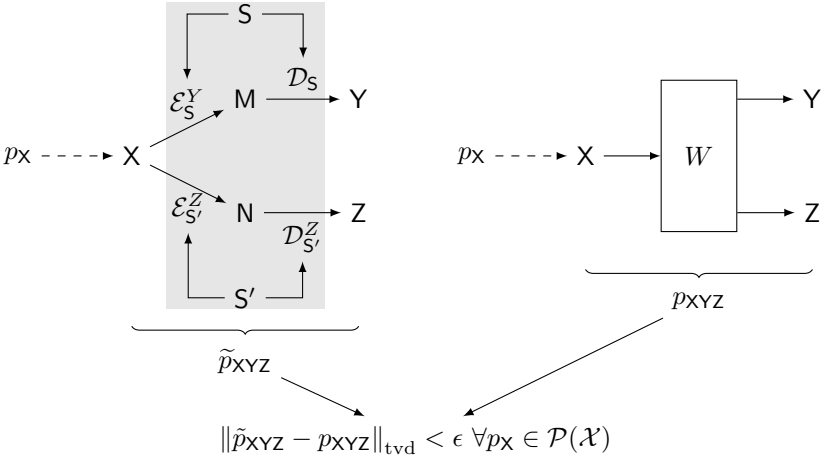
$$C^{\mathrm{NS}}(W,\epsilon) = \inf \left\lceil \sum_{y\in\mathcal{Y}} \zeta(y) \right\rceil \text{ s.t. } \widetilde{W}_{\mathsf{Y}|\mathsf{X}} \in \mathcal{P}(\mathcal{Y}|\mathcal{X}),\ \left\| \widetilde{W}_{\mathsf{Y}|\mathsf{X}} - W_{\mathsf{Y}|\mathsf{X}} \right\|_{\mathrm{tvd}} \leqslant \epsilon,\ \widetilde{W}_{\mathsf{Y}|\mathsf{X}}(y|x) \leqslant \zeta(y)$$





Main Results

Broadcast channel simulation



Main Results

Broadcast channel simulation

$$\mathcal{M}_{\epsilon}^{\text{in}}(W) := \left\{ (M,N) \in \mathbb{Z}_{>0}^2 \;\middle|\; \begin{array}{l} \log M > D_{s+}^{\epsilon_1-\delta_1}\left(p_{\mathsf{X}} \cdot W_{\mathsf{Y}|\mathsf{X}} \middle\| p_{\mathsf{X}} \times q_{\mathsf{Y}}\right) - \log \delta_1^2 \\ \log N > D_{s+}^{\epsilon_2-\delta_2}\left(p_{\mathsf{X}} \cdot W_{\mathsf{Z}|\mathsf{X}} \middle\| p_{\mathsf{X}} \times r_{\mathsf{Z}}\right) - \log \delta_2^2 \\ \log MN > D_{s+}^{\epsilon_3-\delta_3}\left(p_{\mathsf{X}} \cdot W_{\mathsf{YZ}|\mathsf{X}} \middle\| p_{\mathsf{X}} \times q_{\mathsf{Y}} \times r_{\mathsf{Z}}\right) - \log \delta_3^2 \end{array} \quad \forall p_{\mathsf{X}} \in \mathcal{P}(\mathcal{X}) \right\}$$

$$\mathcal{M}_{\epsilon}^{\text{out}}(W) := \left\{ (M,N) \in \mathbb{Z}_{>0}^2 \;\middle|\; \begin{array}{l} \log M \geqslant \inf\limits_{q_{\mathsf{Y}}} \sup\limits_{p_{\mathsf{X}}} D_{s+}^{\frac{\epsilon}{1-\delta_1}}\left(p_{\mathsf{X}} \cdot W_{\mathsf{Y}|\mathsf{X}} \middle\| p_{\mathsf{X}} \times q_{\mathsf{Y}}\right) + \log \delta_1 \\ \log N \geqslant \inf\limits_{r_{\mathsf{Z}}} \sup\limits_{p_{\mathsf{X}}} D_{s+}^{\frac{\epsilon}{1-\delta_2}}\left(p_{\mathsf{X}} \cdot W_{\mathsf{Z}|\mathsf{X}} \middle\| p_{\mathsf{X}} \times r_{\mathsf{Z}}\right) + \log \delta_2 \\ \log MN \geqslant \inf\limits_{q_{\mathsf{Y}}, r_{\mathsf{Z}}} \sup\limits_{p_{\mathsf{X}}} D_{s+}^{\frac{\epsilon}{1-\delta_3}}\left(p_{\mathsf{X}} \cdot W_{\mathsf{YZ}|\mathsf{X}} \middle\| p_{\mathsf{X}} \times q_{\mathsf{Y}} \times r_{\mathsf{Z}}\right) + \log \delta_3 \end{array} \right\}$$

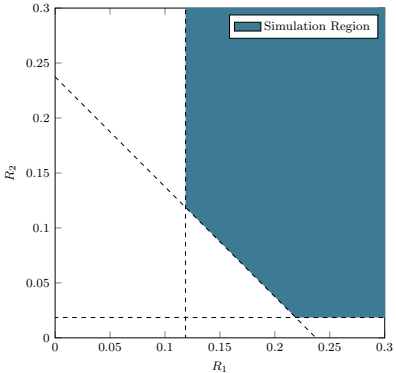
$$D_{s+}^{\epsilon}(p\|q) := \inf\left\{ a \geqslant 0 : P\left[\log \frac{p(\mathsf{X})}{q(\mathsf{X})} > a\right] < \epsilon \right\}$$

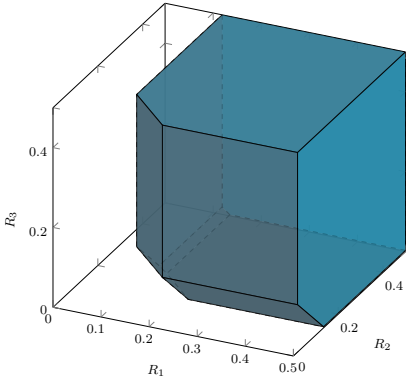
Main Results

Broadcast channel simulation

$$\mathcal{R}_0^\star(W_{\mathsf{YZ}|\mathsf{X}}) := \bigcap_{\epsilon\in(0,1)} \mathcal{R}_\epsilon^\star(W_{\mathsf{YZ}|\mathsf{X}}) = \left\{(r_1, r_2) \in \mathbb{R}^2_{>0} \middle| r_1 \geqslant C(W_{\mathsf{Y}|\mathsf{X}}),\ r_2 \geqslant C(W_{\mathsf{Z}|\mathsf{X}}),\ r_1 + r_2 \geqslant \widetilde{C}(W_{\mathsf{YZ}|\mathsf{X}})\right\}$$

where $\widetilde{C}(W_{\mathsf{YZ}|\mathsf{X}}) := \sup_{p_{\mathsf{X}}\in\mathcal{P}(\mathcal{X})} I(\mathsf{X}:\mathsf{Y}:\mathsf{Z})_{p_{\mathsf{X}}\cdot W_{\mathsf{Y}|\mathsf{X}}}$ with $I(\mathsf{X}:\mathsf{Y}:\mathsf{Z}) := H(\mathsf{X}) + H(\mathsf{Y}) + H(\mathsf{Z}) - H(\mathsf{XYZ})$.





Outline of the talk

In this talk, we will only focus on

- One-shot Achievability Bound $\Leftarrow$ Convex Split Lemma
- One-shot Converse Bound
- Asymptotic Analysis

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An arXiv draft will be ready soon!

Convex Split Lemma(s)

Lemma 1 ([ADJ17], [AJW17, Fact 6])

Let (X, Y) *be jointly distributed over* $\mathcal{X} \times \mathcal{Y}$*, let* q *be some probability distribution over* $\mathcal{Y}$*, and let* $\epsilon > \delta \in (0, 1)$*. Let* M *be a nonnegative integer s.t.*

$$\log M \geqslant D_{\max}^{\epsilon-\delta,\mathsf{X}}(p_{\mathsf{XY}} \| p_{\mathsf{X}} \times q_{\mathsf{Y}}) + \log \frac{2}{\delta^2}, \tag{2}$$

Let J *be uniformly distributed on* $\{1, \ldots, M\}$*, and let the joint random variables* $(\mathsf{J}, \mathsf{X}, \mathsf{Y}_1, \ldots, \mathsf{Y}_M)$ *be distributed as*

$$p_{\mathsf{X},\mathbf{Y}_1^M|\mathsf{J}}(x, \boldsymbol{y}_1^M | j) = p_{\mathsf{XY}}(x, y_j) \cdot \prod_{i \neq j} q(y_i). \tag{3}$$

Then, it holds that

$$\left\| p_{\mathsf{X},\mathsf{Y}_1,\ldots,\mathsf{Y}_M} - p_{\mathsf{X}} \times q_{\mathsf{Y}_1} \times \cdots \times q_{\mathsf{Y}_M} \right\|_{\mathrm{TVD}} \leqslant \epsilon. \tag{4}$$

Convex Split Lemma(s)

Proof Sketch.

1. Let $p_{\mathsf{X}'\mathsf{Y}'}$ be an optimal pmf for the definition of $D_{\max}^{\epsilon-\delta,\mathsf{X}}$, *i.e.*, $\|p_{\mathsf{X}'\mathsf{Y}'} - p_{\mathsf{X}\mathsf{Y}}\| \leqslant \epsilon - \delta$, $\log \frac{p_{\mathsf{X}'\mathsf{Y}'}(x,y)}{p(x)\cdot q(y)} \leqslant D_{\max}^{\epsilon-\delta}(p_{\mathsf{X}\mathsf{Y}} \| p_{\mathsf{X}} \times q_{\mathsf{Y}})$ and $p_{\mathsf{X}'} = p_{\mathsf{X}}$.
2. Construct random variables $(\mathsf{J}', \mathsf{X}', \mathsf{Y}_1, \ldots, \mathsf{Y}_M)$ from $(\mathsf{X}', \mathsf{Y}')$ in the same fashion as the construction of $(\mathsf{J}, \mathsf{X}, \mathsf{Y}_1, \ldots, \mathsf{Y}_M)$ from (X, Y).
3. Using properties of $D(\cdot\|\cdot)$, one can conclude that

$$D\left(p_{\mathsf{X}'\mathsf{Y}'_1,\ldots,\mathsf{Y}'_M} \middle\| p_{\mathsf{X}'} \cdot \prod_{i=1}^{M} q_{\mathsf{Y}_i}\right) \leqslant \sum_{x,y} p_{\mathsf{X}'\mathsf{Y}'}(x,y) \cdot \log\left(1 + \frac{1}{M}\frac{p_{\mathsf{X}'\mathsf{Y}'}(x,y)}{p_{\mathsf{X}'}(x)\cdot p_{\mathsf{Y}'}(y)}\right)$$

$$\leqslant \frac{1}{M} \cdot \max_{x,y}\left\{\frac{p_{\mathsf{X}'\mathsf{Y}'}(x,y)}{p_{\mathsf{X}'}(x)\cdot p_{\mathsf{Y}'}(y)}\right\} \leqslant \frac{\delta^2}{2}$$

One-Shot Achievability Bound

Theorem 2

Let W be a DMC from $\mathcal{X}$ to $\mathcal{Y}$, and let $\epsilon \in (0,1)$. A size-M ϵ-simulation code for DMC W exists if

$$\log M > \inf_{q_{\mathsf{Y}}} \sup_{p_{\mathsf{X}}} D_{\max}^{\epsilon-\delta,\mathsf{X}}\left(p_{\mathsf{X}} \cdot W_{\mathsf{Y}|\mathsf{X}} \middle\| p_{\mathsf{X}} \times q_{\mathsf{Y}}\right) + \log \frac{2}{\delta^2}, \tag{5}$$

for some $\delta \in (0,\epsilon)$.

Convex Split Lemma: when M is large enough ...

$$\begin{tikzcd}
\sum_{j=1}^{M} \frac{1}{M} p_{\mathsf{XY}}(x,y_j) \cdot \prod_{i\neq j} q(y_i) \arrow[r, leftrightarrow, "\epsilon"] \arrow[d, "p_{\mathsf{J}|\mathsf{X},\mathsf{Y}_1^M}(j|x,\mathbf{y}_1^M) := \frac{W(y_j|x)\cdot\prod_{\ell\neq j} q(y_\ell)}{\sum_{i=1}^{M} W(y_i|x)\cdot\prod_{\ell\neq i} q(y_\ell)}"'] & p_{\mathsf{X}'}(x) \cdot \prod_{i=1}^{M} q_{\mathsf{Y}_i}(y_j) \arrow[d, "\text{same conditional pmf as LHS}"'] \\
\frac{1}{M} p_{\mathsf{XY}}(x,y_j) \cdot \prod_{i\neq j} q(y_i) \arrow[r, leftrightarrow, "\epsilon"] & \mathsf{J}', \mathsf{X}', \mathsf{Y}'_1, \ldots, \mathsf{Y}'_M
\end{tikzcd}$$

One-Shot Achievability Bound

Proof Idea.

1. Let q_{Y} be a pmf on $\mathcal{Y}$ such that

$$\log M > \sup_{p_{\mathsf{X}}} D_{\max}^{\epsilon-\delta,\mathsf{X}}\left(p_{\mathsf{X}} \cdot W_{\mathsf{Y}|\mathsf{X}} \middle\| p_{\mathsf{X}} \times q_{\mathsf{Y}}\right) + \log \frac{2}{\delta^2}.$$

1. Let Alice and Bob share some common i.i.d. random variables $(\mathsf{Y}_1, \ldots, \mathsf{Y}_M)$ where $\mathsf{Y}_k \sim q$ for each k.
2. Upon receiving input $\mathsf{X} = x$, Alice generates a random variable J (distributed on $\{1, \ldots, M\}$) according to the following conditional pmf

$$\widetilde{p}_{\mathsf{J}|\mathsf{X},\mathsf{Y}_1^M}(j|x, \boldsymbol{y}_1^M) := \frac{W(y_j|x) \cdot \prod_{\ell \neq j} q(y_\ell)}{\sum_{i=1}^M W(y_i|x) \cdot \prod_{\ell \neq i} q(y_\ell)}.$$

3. Alice sends J losslessly to Bob using $\log M$-bits.
4. Upon receiving J, Bob output Y_{J}.

One-Shot Converse Bound

Theorem 3

Let W be a DMC from $\mathcal{X}$ to $\mathcal{Y}$, and let $\epsilon \in (0,1)$. If a size-M ϵ-simulation code for W exists, then

$$\log M \geqslant \inf_{q_{\mathsf{Y}}} \sup_{p_{\mathsf{X}}} D_{\max}^{\epsilon,\mathsf{X}}(p_{\mathsf{X}} \cdot W_{\mathsf{Y}|\mathsf{X}} \| p_{\mathsf{X}} \times q_{\mathsf{Y}}). \tag{6}$$

Proof Sketch.

1. Let XY be distributed according to the simulation protocol as $\widetilde{p}_{\mathsf{XY}}$ $(\mathsf{X} - \mathsf{MS} - \mathsf{Y})$;
1. Lockability ([Ber13, Corollary A.14]): $\log M = I_{\max}(\mathsf{X};\mathsf{S}) + \log M \geqslant I_{\max}(\mathsf{X};\mathsf{MS})$;
2. DPI of $I_{\max}$: $I_{\max}(\mathsf{X};\mathsf{MS}) \geqslant I_{\max}(\mathsf{X};\mathsf{Y})$;
3. By definition of the smoothed max-divergence, and noting that $\widetilde{p}_{\mathsf{X}} = p_{\mathsf{X}}$:

$$I_{\max}(\mathsf{X};\mathsf{Y}) := \inf_{q_{\mathsf{Y}}} D_{\max}(\widetilde{p}_{\mathsf{XY}} \| p_{\mathsf{X}} \times q_{\mathsf{Y}}) = \inf_{q_{\mathsf{Y}}} \max_{x,y} \log \frac{\widetilde{p}_{\mathsf{Y}|\mathsf{X}}(y|x)}{q_{\mathsf{Y}}(y)}$$
$$= \inf_{q_{\mathsf{Y}}} \sup_{p_{\mathsf{X}}} D_{\max}(\widetilde{p}_{\mathsf{XY}} \| p_{\mathsf{X}} \times q_{\mathsf{Y}}) \geqslant \inf_{q_{\mathsf{Y}}} \sup_{p_{\mathsf{X}}} D_{\max}^{\epsilon,\mathsf{X}}(p_{\mathsf{XY}} \| p_{\mathsf{X}} \times q_{\mathsf{Y}}).$$

Asymptotic Analysis of the Achievability Bound

$$\limsup_{n\to\infty} \frac{1}{n} \log M_n^\star(W^{\otimes n}) \leqslant C(W) \tag{7}$$

Proof Sketch.

$$\begin{aligned}
\frac{1}{n}\log M_n^\star &\leqslant \inf_{q_{\mathbf{Y}_1^n}} \sup_{p_{\mathbf{X}_1^n}} \frac{1}{n} D_{\max}^{\epsilon-\delta,\mathsf{X}}\left(p_{\mathbf{X}_1^n}\cdot W_{\mathsf{Y}|\mathsf{X}}^{\otimes n}\middle\| p_{\mathbf{X}_1^n}\times q_{\mathbf{Y}_1^n}\right) + \frac{1}{n}\log\frac{2}{\delta^2} && \text{Theorem 2}\\
&\leqslant \inf_{q_{\mathbf{Y}_1^n}} \sup_{p_{\mathbf{X}_1^n}} \frac{1}{n} D_{s+}^{\epsilon-\delta}\left(p_{\mathbf{X}_1^n}\cdot W_{\mathsf{Y}|\mathsf{X}}^{\otimes n}\middle\| p_{\mathbf{X}_1^n}\times q_{\mathbf{Y}_1^n}\right) + \frac{1}{n}\log\frac{4}{\delta^2} && \text{[ABJT20, Thm 9]}
\end{aligned}$$

$$D_{s+}^{\epsilon}(p\|q) := \inf\left\{a \geqslant 0 : P\left[\log\frac{p(\mathsf{X})}{q(\mathsf{X})} > a\right] < \epsilon\right\}. \tag{8}$$

Theorem 4 (Modified version of [ABJT20, Thm 9])

$$D_{\max}^{\epsilon,\mathsf{X}}(p_{\mathsf{XY}}\|p_{\mathsf{X}}\times q_{\mathsf{Y}}) \leqslant D_{s+}^{\epsilon}(p_{\mathsf{XY}}\|p_{\mathsf{X}}\cdot q_{\mathsf{Y}}) + 1. \tag{9}$$

Asymptotic Analysis of the Achievability Bound

Proof Sketch (Continue).

$$\cdots \leqslant \inf_{q_{\mathbf{Y}_1^n}} \sup_{\boldsymbol{x}_1^n} \frac{1}{n} D_{s+}^{\epsilon-\delta}\left(W_{\mathsf{Y}|\mathsf{X}}^{\otimes n}(\cdot|\boldsymbol{x}_1^n)\middle\| q_{\mathbf{Y}_1^n}\right) + \frac{1}{n}\log\frac{4}{\delta^2} \qquad \text{Lemma 5}$$

$$\overset{\star}{\leqslant} \sup_{\boldsymbol{x}_1^n} \frac{1}{n} D_{s+}^{\epsilon-\delta}\left(W_{\mathsf{Y}|\mathsf{X}}^{\otimes n}(\cdot|\boldsymbol{x}_1^n)\middle\| \sum_{\lambda\in\Lambda_n}\frac{1}{|\Lambda_n|}\left(\sum_{\widetilde{x}} W_{\mathsf{Y}|\mathsf{X}}(\cdot|\widetilde{x})\cdot p_\lambda(\widetilde{x})\right)^{\otimes n}\right) + \frac{1}{n}\log\frac{4}{\delta^2}$$

Lemma 5

$$D_{s+}^{\epsilon}\left(p_{\mathsf{Y}|\mathsf{X}}\cdot p_{\mathsf{X}}\middle\| q_{\mathsf{Y}|\mathsf{X}}\cdot p_{\mathsf{X}}\right) \leqslant \sup_{x\in\mathcal{X}} D_{s+}^{\epsilon}\left(p_{\mathsf{Y}|\mathsf{X}}(\cdot|x)\middle\| q_{\mathsf{Y}|\mathsf{X}}(\cdot|x)\right). \tag{10}$$

$\star$

- $q_{\mathbf{Y}_1^n}(\boldsymbol{y}^n) := \sum_{\lambda\in\Lambda_n}\frac{1}{|\Lambda_n|}\prod_{i=1}^n\sum_{\widetilde{x}} W(y_i|\widetilde{x})p_\lambda(\widetilde{x})$;
- $\Lambda_n := \left\{\lambda\subset\mathcal{X}^n : \sum_{i=1}^n \delta_{x,x_i} = \sum_{i=1}^n \delta_{x,\widetilde{x}_i}\forall \boldsymbol{x}_1^n, \widetilde{\boldsymbol{x}}_1^n\in\lambda, \forall x\in\mathcal{X}\right\}$;
- p_λ is a pmf on $\mathcal{X}$ such that $p_\lambda(\widetilde{x})$ equals the frequency of $\widetilde{x}$ in sequence $\boldsymbol{x}\in\lambda$.

Asymptotic Analysis of the Achievability Bound

Proof Sketch (Continue).

$$\cdots \leqslant \sup_{\boldsymbol{x}_1^n} \frac{1}{n} D_{s+}^{\epsilon-\delta}\left(W_{\mathsf{Y}|\mathsf{X}}^{\otimes n}(\cdot|\boldsymbol{x}_1^n) \middle\| \left(\sum_{\widetilde{x}} W_{\mathsf{Y}|\mathsf{X}}(\cdot|\widetilde{x}) \cdot f_{\boldsymbol{x}_1^n}(\widetilde{x})\right)^{\otimes n}\right) \qquad \text{[TT13, Lem 3]}$$

$$+\frac{1}{n}\log|\Lambda_n| + \frac{1}{n}\log\frac{4}{\delta^2}$$

$$\leqslant \sup_{\boldsymbol{x}_1^n} \left\{\frac{1}{n}\sum_{i=1}^{n} D\left(W_{\mathsf{Y}|\mathsf{X}}(\cdot|x_i) \middle\| \sum_{\widetilde{x}} W_{\mathsf{Y}|\mathsf{X}}(\cdot|\widetilde{x}) \cdot f_{\boldsymbol{x}_1^n}(\widetilde{x})\right) + \right. \qquad \text{[TT13, Lem 5]}$$

$$\left. \frac{1}{\sqrt{n(\epsilon-\delta)}} \cdot \sqrt{\frac{1}{n}\sum_{i=1}^{n} V\left(W_{\mathsf{Y}|\mathsf{X}}(\cdot|x_i) \middle\| \sum_{\widetilde{x}} W_{\mathsf{Y}|\mathsf{X}}(\cdot|\widetilde{x}) \cdot f_{\boldsymbol{x}_1^n}(\widetilde{x})\right)}\right\}$$

$$+\frac{1}{n}\log|\Lambda_n| + \frac{1}{n}\log\frac{4}{\delta^2},$$

Asymptotic Analysis of the Achievability Bound

Proof Sketch (Continue).

$$\cdots = \frac{1}{n}\log\frac{4\,|\Lambda_n|}{\delta^2} + \sup_{\boldsymbol{x}_1^n}\Bigg\{ I(\mathsf{X};\mathsf{Y})_{f_{\boldsymbol{x}_1^n}\cdot W_{\mathsf{Y}|\mathsf{X}}} + \qquad \text{by counting}$$

$$\frac{1}{\sqrt{n(\epsilon-\delta)}} V\left(f_{\boldsymbol{x}_1^n}\cdot W_{\mathsf{Y}|\mathsf{X}} \,\middle\|\, \sum_{\widetilde{x}} f_{\boldsymbol{x}_1^n}(\widetilde{x})\cdot W_{\mathsf{Y}|\mathsf{X}}(\cdot|\widetilde{x}) \right)\Bigg\}$$

$$\leqslant C(W) + \underbrace{\frac{1}{n}\log\frac{4\,|\Lambda_n|}{\delta^2}}_{\downarrow\ 0} + \frac{1}{\sqrt{n(\epsilon-\delta)}} \sup_{\boldsymbol{x}_1^n}\Bigg\{ \underbrace{V\left(f_{\boldsymbol{x}_1^n}\cdot W_{\mathsf{Y}|\mathsf{X}} \,\middle\|\, \sum_{\widetilde{x}} f_{\boldsymbol{x}_1^n}(\widetilde{x})\cdot W_{\mathsf{Y}|\mathsf{X}}(\cdot|\widetilde{x}) \right)}_{\downarrow\ 0}\Bigg\}$$

Asymptotic Analysis of the Converse Bound

$$\liminf_{n\to\infty} \frac{1}{n} \log M_n^\star(W^{\otimes n}) \geqslant C(W) \tag{11}$$

Proof Sketch.

$$\frac{1}{n}\log M_n^\star \geqslant \frac{1}{n} \sup_{p_{\mathbf{X}_1^n}} \inf_{q_{\mathbf{Y}_1^n}} D_{\max}^{\epsilon}(p_{\mathbf{X}_1^n} \cdot W_{\mathsf{Y}|\mathsf{X}}^{\otimes n} \| p_{\mathbf{X}_1^n} \times q_{\mathbf{Y}_1^n}) \qquad \text{Theorem 6, relaxed}$$

$$\geqslant \frac{1}{n} \sup_{p_{\mathbf{X}_1^n}} \inf_{q_{\mathbf{Y}_1^n}} D_{s+}^{\frac{\epsilon}{1-\delta}}(p_{\mathbf{X}_1^n} \cdot W_{\mathsf{Y}|\mathsf{X}}^{\otimes n} \| p_{\mathbf{X}_1^n} \times q_{\mathbf{Y}_1^n}) + \frac{\log \delta}{n} \qquad \text{Lemma 6}$$

Lemma 6 ([ABJT20, Eq. (48)])

$$D_{\max}^{\epsilon,\mathsf{X}}(p_{\mathsf{XY}} \| p_{\mathsf{X}} \times q_{\mathsf{Y}}) \geqslant D_{s+}^{\frac{\epsilon}{1-\delta}}(p_{\mathsf{XY}} \| p_{\mathsf{X}} \times q_{\mathsf{Y}}) + \log \delta. \tag{12}$$

Asymptotic Analysis of the Converse Bound

Proof Sketch (Countinue).

$$\cdots \geqslant \frac{1}{n} \sup_{p_{\mathbf{X}_1^n}} D_{s+}^{\frac{\epsilon}{1-\delta}+\gamma}(p_{\mathbf{X}_1^n} \cdot W_{\mathsf{Y}|\mathsf{X}}^{\otimes n} \| p_{\mathbf{X}_1^n} \times p_{\mathbf{Y}_1^n}) + \frac{\log \delta + \log \gamma}{n} \qquad \text{Lemma 7}$$

$$\geqslant \underbrace{\frac{1}{n} \sup_{p_{\mathsf{X}}} D_{s+}^{\frac{\epsilon}{1-\delta}+\gamma}((p_{\mathsf{X}} \cdot W_{\mathsf{Y}|\mathsf{X}})^{\otimes n} \| (p_{\mathsf{X}} \times p_{\mathsf{Y}})^{\otimes n})}_{\downarrow\, C(W)} + \underbrace{\frac{\log \delta + \log \gamma}{n}}_{\downarrow\, 0} \qquad \text{restricting to i.i.d.}$$

Lemma 7 ([ABJT20, Lemma 10])

$$\inf_{q_{\mathsf{Y}}} D_{s+}^{\epsilon}(p_{\mathsf{XY}} \| p_{\mathsf{X}} \times q_{\mathsf{Y}}) \geqslant D_{s+}^{\epsilon+\delta}(p_{\mathsf{XY}} \| p_{\mathsf{X}} \times p_{\mathsf{Y}}) + \log \delta. \tag{13}$$

First Order Result

Combining the two asymptotic analysis, we know $\lim_{n\to\infty} \frac{1}{n} \log M_{\epsilon}^{\star}(W^{\otimes n}) = C(W)$.

Summary

- The cost for channel simulation with common randomness can be described by $I^{\epsilon}_{\max}$, *i.e.*,

$$I^{\epsilon}_{\max}(W_{\mathsf{Y}|\mathsf{X}}) \leqslant \log M^{\star}_{\epsilon}(W_{\mathsf{Y}|\mathsf{X}}) \leqslant I^{\epsilon-\delta}_{\max}(\mathsf{X}:\mathsf{Y})_{W_{\mathsf{Y}|\mathsf{X}}} + \log \frac{4}{\delta^2} \quad \text{for any} \quad 0 < \delta \leqslant \epsilon$$

- The achievability is a result of convex split lemma.
- The converse is a result of lockability and DPI of $I_{\max}$ (among other things).
- Asymptotically, the cost coincide with the capacity of the channel(, and we used the information spectrum method for the analysis). This implies channel interconversion on the first order.

Noteworthy, but not presented today

Higher-Order Expansions

Let W be a DMC. For each $\epsilon \in (0, 1)$ such that $V_\epsilon = 0$, it holds that

$$\log M_\epsilon^\star(W^\otimes) = nC(W) + \sqrt{nV_\epsilon(W)} \cdot \Phi^{-1}(1-\epsilon) + \Theta(\log n). \tag{14}$$

In the case when $V_\epsilon = 0$, it holds that

$$\log M_\epsilon^\star(W^{\otimes n}) \leqslant nC(W) + O(1) \tag{15}$$

if $\epsilon \in (0, \frac{1}{2}]$, and W *not* being exotic ; or $\epsilon \in (\frac{1}{2}, 1)$.

Moderate-Deviation Analysis

Let W be a DMC, and let $\{a_n\}_n \in \mathbb{N}$ be a moderate sequence. It holds that

$$\log M_{\epsilon_n}^\star(W^\otimes) = nC(W) + \sqrt{V_{\max}(W)} \cdot a_n \tag{16}$$

where $\epsilon_n := 2^{-na_n^2}$.